ECO239

Week 10

Practice

- 40% of students admitted to university A will actually enroll.

a. What is the probability that at most 1 student will enroll if the college offers admission to 5 students?

$$P(X) = \frac{n!}{x!\,(n-x)!} P^{x}(1-P)^{n-x}$$

a. What is the probability that at most 1 student will enroll if the college offers admission to 5 students?

$$P(X) = \frac{n!}{x!\,(n-x)!}P^x(1-P)^{n-x}$$

- n=5
- p=0.4
- X: at most 1 ($X \leq 1$)

- $P(X \leq 1|n=5,p=0.4) = P(X=0)+P(X=1)$
- =[5!/(0!5!)]*(0.4^0)*(0.6^5)
- +[5!/(1!4!)]*(0.4^1)*(0.6^4)
- =0.337

b. What is the probability that between 2 and 4 students (including 2 and 4) will enroll if the college offers admissions to 5 students?

$$P(X) = \frac{n!}{x!\,(n-x)!}P^x(1-P)^{n-x}$$

- n=5
- p=0.4
- X: between 2 and 4 (including 2 and 4)

- $P(2 \leq X \leq 4|n=5, p=0.4)$
- =P(X=2)+P(X=3)+P(X=4)
- =5!/(2!3!)*(0.4)^2*(0.6)^3
- +5!/(3!2!)*(0.4)^3*(0.6)^2
- +5!/(4!1!)*(0.4)^4*(0.6)^1
- =0.653

c. What is the probability that at most 6 students will enroll if the college offers admission to 10 students?

- $P(X \leq 6|n=10,p=0.4))$
- $=P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)$
- …

⇒Use "Cumulative Binomial Probabilities (Table 3)" table for this calculation.

- By using the table, look for n = 10, x = 6, P=0.4 and read the corresponding $P(X \leq 6|n=10,p=0.4)= 0.945$.

d. What is the probability that more than 9 will enroll if 13 students was offered the admissions?

- $P(X>9|n=13, p=0.4)$

- How can we solve this probability???

- $= 1- P(X \leq 9|n=13,p=0.4)$
- $=1-0.992 = 0.008$

- Since more than 9 will enroll = 10,11,12 or 13 students will enroll = 1 – P(0~9 students will enroll)

X: Number of Success (Enrollment)
0
1
2
3
4
5
6
7
8
9
10
11
12
13

= 1- P(At most 9 OR Less than 10)

More than 9
(or At least 10)

If 70% of these students admitted actually enroll, what is the probability that at least 8 out of 13 students will actually enroll?

- $P(X \geq 8 | n=13, p=0.7)$

- How can we solve this problem?
- We do not have the table with $P>0.5$.

$\Rightarrow$Redefine the definition of p as "probability of non-enrollment".
- $P(X=8,9,10,11,12$ or 13 enrolling$|\ n=13,\ p=0.7)$
- $=P(X=5,4,3,2,1$ or 0 not enrolling$|\ n=13, p=0.3)$
- $=P(X \leq 5|\ n = 13,\ p=0.3) =$ by using the table

X: Number of Success (Enrollment) P(X)=0.7	X' : Number of Failure (Non-Enrollment) P(X')=0.3
0	13
1	12
2	11
3	10
4	9
5	8
6	7
7	6
8	5
9	4
10	3
11	2
12	1
13	0

In summary...

- At least a = $P(X \geq a)$
- At most a = $P(X \leq a)$
- Fewer than a = $P(X<a)$
- More than a = $P(X>a)$

- In order to use the table,
- $P \leq 0.5$ and $P(X \leq a)$

- Otherwise...

- If $p > 0.5$, then switch “success” and “failure” definitions and use “failure” probability.
- $P(X \geq a | n, p = p_0)$
- $=P(X \leq n-a | n, p = (1-p_0))$

- If $P(X>a|n, p)$, then use
- $1-P(X \leq a|n, p)$

- If $P(X \geq a|n, p)$, then use
- $1-P(X<a|n, p)$
- $=1-P(X \leq a-1|n, p)$

- If $P(X<a|n, p)$ then use
- $P(X \leq a-1|n,p)$

Shapes of binomial distributions

Rcmdr => Distribution=> Discrete => Binomial => Plot binomial distribution.
Set x, n, p.

- Set the number of trials to 20 and the probability of success to 0.15. Describe the shape of the distribution of number of successes.

- What happens to the shape of the distribution as *n* stays constant and *p* changes?

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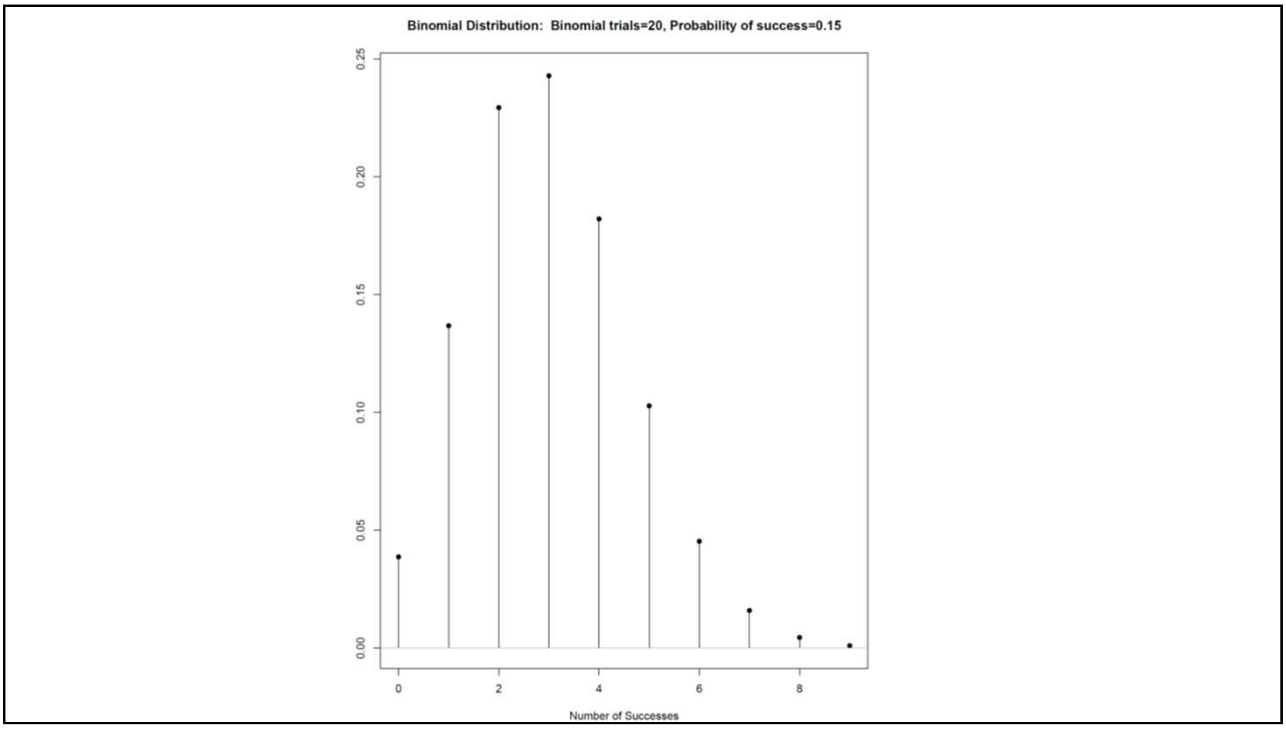
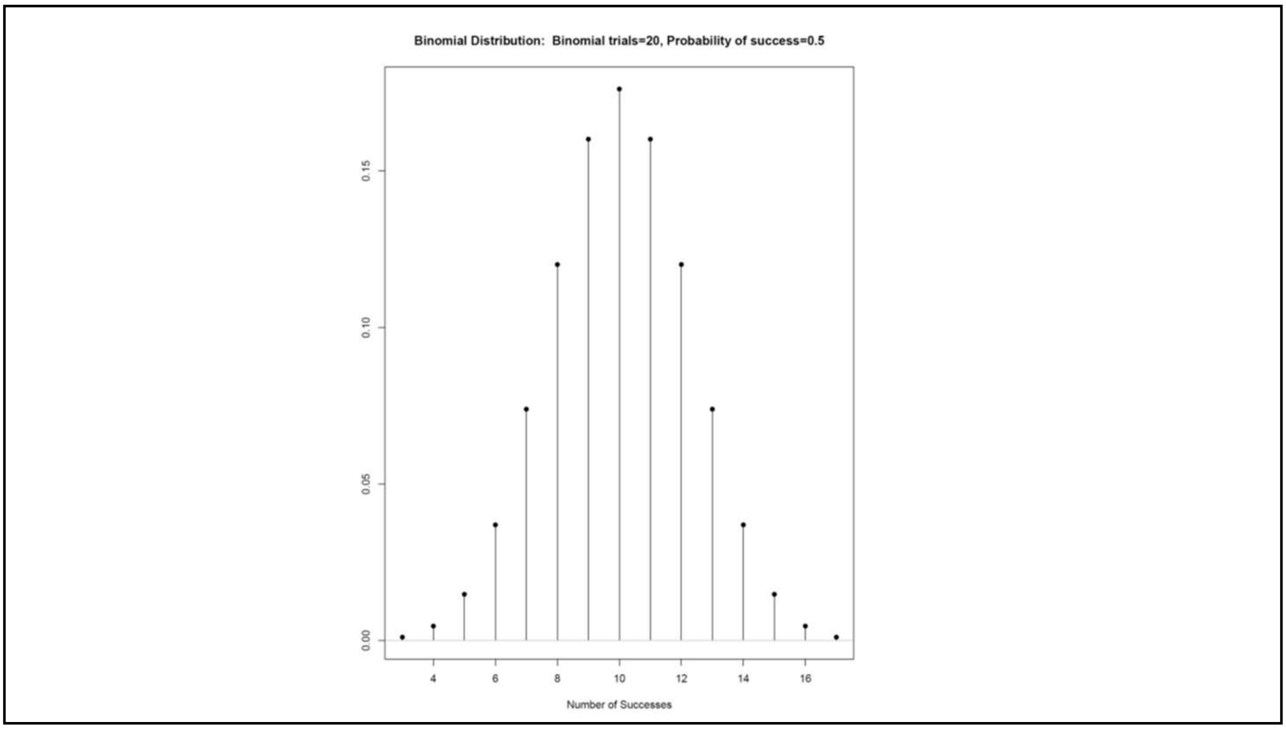
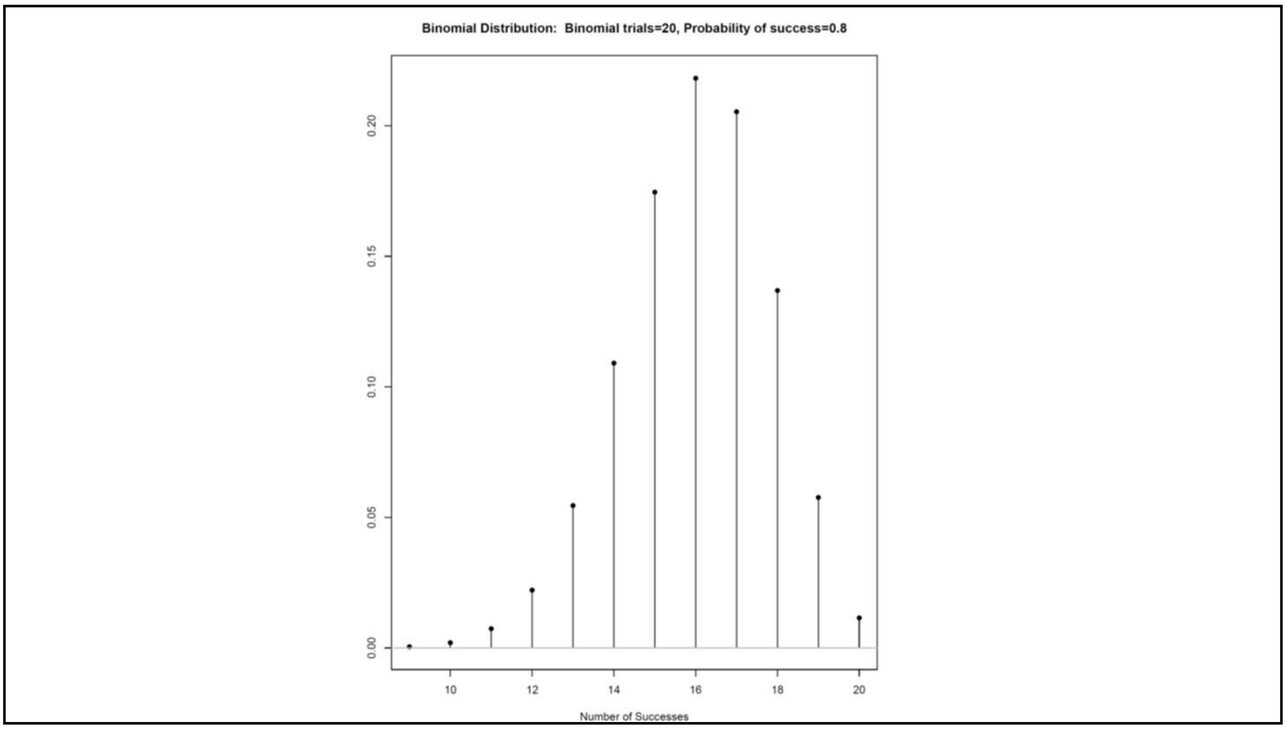
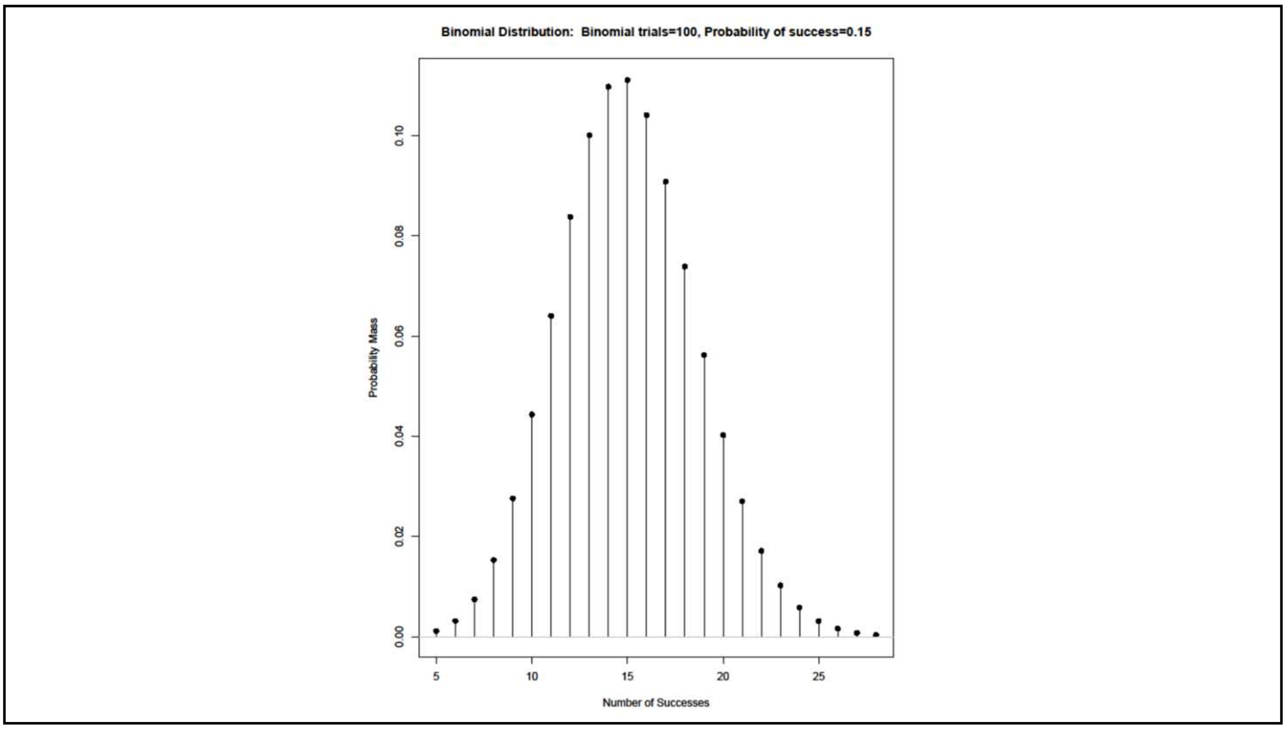


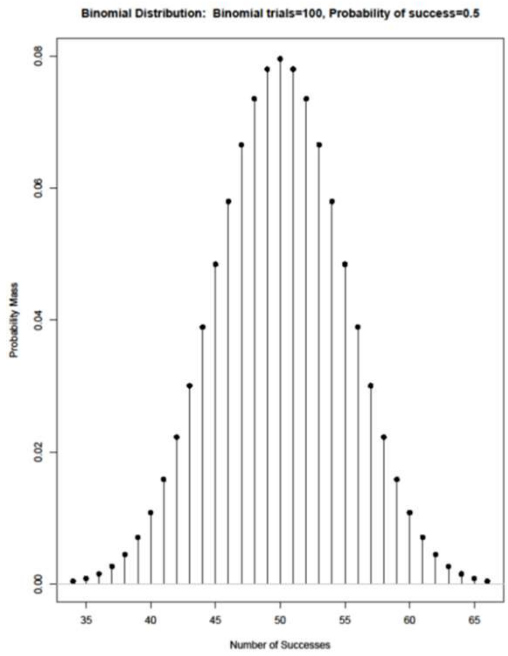
Binomial Distribution: Binomial trials=20, Probability of success=0.15
0.00
0.05
0.10
0.15
0.20
0.25
0
2
4
6
8
Number of Successes

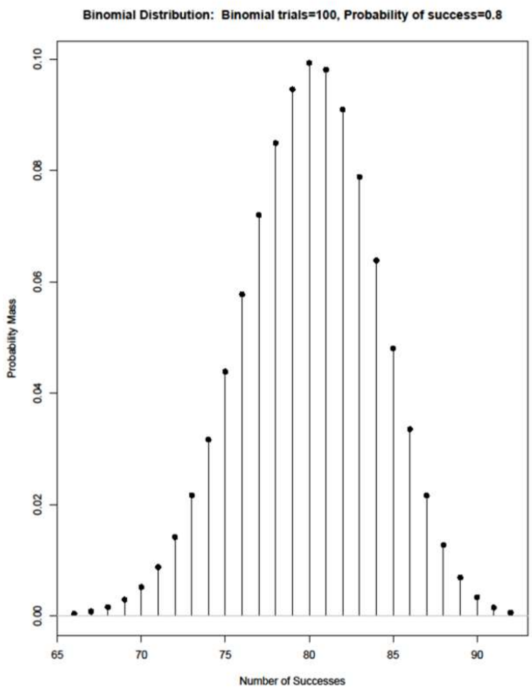


Binomial Distribution: Binomial trials=20, Probability of success=0.5
0.00
0.05
0.10
0.15
4
6
8
10
12
14
16
Number of Successes


Binomial Distribution: Binomial trials=20, Probability of success=0.8
0.00
0.05
0.10
0.15
0.20
10
12
14
16
18
20
Number of Successes

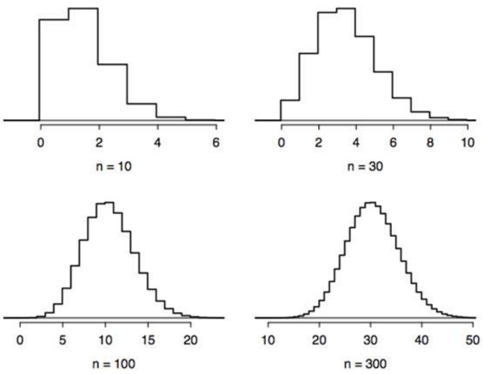

Binomial Distribution: Binomial trials=100, Probability of success=0.15
Probability Mass
0.00
0.02
0.04
0.06
0.08
0.10
5
10
15
20
25
Number of Successes


Binomial Distribution: Binomial trials=100, Probability of success=0.5
Probability Mass
0.00
0.02
0.04
0.06
0.08
35
40
45
50
55
60
65
Number of Successes


Binomial Distribution: Binomial trials=100, Probability of success=0.8
Probability Mass
0.00
0.02
0.04
0.06
0.08
0.10
65
70
75
80
85
90
Number of Successes

Distributions of number of successes

Histograms of samples from the binomial model where $p = 0.10$ and $n = 10, 30, 100,$ and 300. What happens as n increases?



How large is large enough?

The sample size is considered large enough if the expected number of successes and failures are both at least 10.

$np \geq 10$ and $n(1 - p) \geq 10$

How large is large enough?

The sample size is considered large enough if the expected number of successes and failures are both at least 10.

$np \geq 10$ and $n(1 - p) \geq 10$

OR, $n > 10/p$ and $n>10/(1-p)$

For example, If $p = 0.13$, n should be greater than $10/0.13 \approx 77$

$100 \times 0.13 \approx 13$

$100 \times (1 - 0.13) = 87$

Practice

Below are four pairs of Binomial distribution parameters. Which distribution can be approximated by the normal distribution?

A. $n = 100$, $p = 0.95$
B. $n = 25$, $p = 0.45$
C. $n = 150$, $p = 0.05$
D. $n = 500$, $p = 0.015$

Practice

Below are four pairs of Binomial distribution parameters. Which distribution can be approximated by the normal distribution?

A. $n = 100$, $p = 0.95$
B. $n = 25$, $p = 0.45 \rightarrow 25 \times 0.45 = 11.25$, $25 \times 0.55 = 13.75$
C. $n = 150$, $p = 0.05$
D. $n = 500$, $p = 0.015$

Hypergeometric Distribution

- “n” trials in a sample taken from a finite population of size N.
- Samples taken without replacement
- Outcome & trials are dependent
- Concerned with finding the probability of “X” success in the sample where there are “S” successes in the population

Hypergeometric Distribution

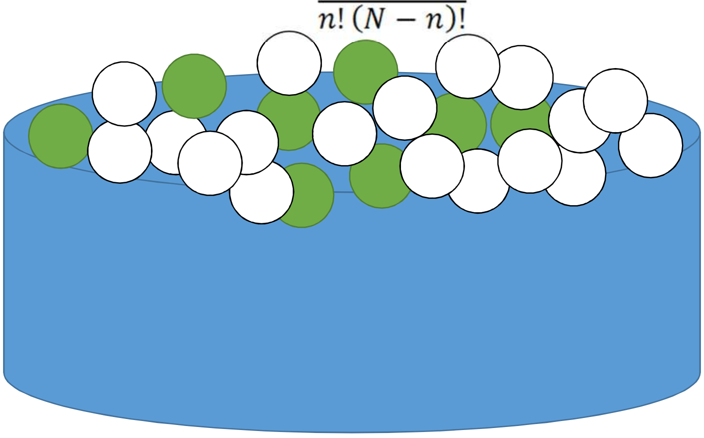
$$P(X) = \frac{C_X^S C_{n-X}^{N-S}}{C_n^N}$$

$$= \frac{\frac{S!}{X!\,(S-X)!}\frac{(N-S)!}{(n-X)!\,(N-S-n+X)!}}{\frac{N!}{n!\,(N-n)!}}$$

where
N = population size
S = # of success in the population
N-S = # of failure in the population
n = sample size
X = # of successes in the sample
n-X = # of failures in the sample.

$$P(X) = \frac{C_X^S C_{n-X}^{N-S}}{C_n^N}$$

$$= \frac{\frac{S!}{X!\,(S-X)!}\frac{(N-S)!}{(n-X)!\,(N-S-n+X)!}}{\frac{N!}{n!\,(N-n)!}}$$



Success: Choosing a red ball, Failure: Choosing a white ball

- C_X^S : # of possible ways that X success can be selected for the sample out of S successes contained in the population.

- C_{n-X}^{N-S}: (N-S) failures can be selected for the sample out of (N-S) failures in the population.

- C_n^N: total number of different sample size n that can be obtained from the population size N.

Practice

$$P(X) = \frac{C_X^S \, C_{n-X}^{N-S}}{C_n^N}$$

3 different computers are checked from 10 in the department. 4 of the 10 computers have illegal software loaded. What is the probability that 2 of the 3 selected computers have illegal software loaded?

N=10
S=4
n=3
X=2

$$P(X) = \frac{C_2^4 \, C_{3-2}^{10-4}}{C_3^{10}}$$

$$= \frac{\frac{4!}{2!\,(4-2)!}\,\frac{(10-4)!}{(3-2)!\,(10-4-3+2)!}}{\frac{10!}{3!\,(10-3)!}}$$

$$= \frac{\frac{4!}{2!\,2!}\,\frac{6!}{1!\,5!}}{\frac{10!}{3!\,7!}} = \frac{6*6}{120} = 0.3$$

Probability of selecting 2 of the 3 selected computers have illegal software loaded is 30%.

Practice

A company receives a shipment of 16 items. A random sample of 4 items is selected and the shipment is rejected if any of these items proves to be defective.

Q1. What is the probability of accepting a shipment containing 4 defective items?

N=16, S=4, n=4, X=0
P(X=0)= 0.2720

Q2. What is the probability of accepting a shipment containing 1 defective item?

N=16, S=1, n=4, x=0
P(X=0)=0.75

Q3. What is the probability of rejecting a shipment containing 1 defective item?

N=16, S=1, n=4, X= ?

1-P(X=0)
=1-0.75
=0.25

A company receives a shipment of 12 items. A random sample of 4 items is selected and the shipment is rejected if 2 or more of these items proves to be defective.

Q. What is the probability of accepting a shipment containing 5 defective items?

P(X=0)+P(X=1)

Poisson Distribution

Apply the Poisson Distribution when
- You wish to **count the number of times** an event occurs in a given continuous interval.
- The probability that an event occurs in one subinterval is very small and is the same for all subintervals.
- The number of events that occur in one subinterval is independent of the number of events that occur in the other subintervals.

e.g. bus arrivals, customer arrivals, phone calls…

Poisson Distribution

$$P(X) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

- X: # of successes over a given time/space
- λ : expected number of successes per given time/space.

Should be the same unit!

Poisson Distribution

- Mean
- Variance

$$\mu = E(X) = \lambda \qquad \sigma^2 = E[(x-\mu)^2] = \lambda$$

Practice

$$P(X) = \frac{e^{-\lambda}\lambda^x}{x!}$$

3 component failures occurred during last 100 days in his computer system.

a. What is the probability of no failure in a given day?

   What is λ ? $\lambda = 3/100 = 0.03$ per day.

$$P(X = 0|\lambda = 0.03) = \frac{e^{-0.03}0.03^0}{0!} = e^{-0.03} = 0.9704$$

Practice

$$P(X) = \frac{e^{-\lambda}\lambda^x}{x!}$$

b. What is the probability of one or more components failures in a given day?

1-P(X=0|λ=0.03) =0.029554

c. What is the probability of at least 2 failures in a 3-day period?

What is λ ?
λ = 0.03*3 = 0.09/3 days

$$\begin{aligned} &P(X \geq 2|\lambda = 0.09) \\ &= 1 - P(X \leq 1|\lambda = 0.09) \\ &= 1 - [P(X = 0) + P(X = 1) \\ &\quad = 1 - [\frac{e^{-0.09}0.09^0}{0!} \\ &\quad + \frac{e^{-0.09}0.09^1}{1!}] \\ &\quad = 1 - (e^{-0.09}(1 + 0.09)) \\ &\quad = 0.0038 \end{aligned}$$

Practice

$$P(X) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

- Customers arrive at a photocopy center at an average rate of two every 5 minutes.
- Find a probability that more than two customers arrive in a 5-minute period.

X: # of arriving customers in a 5 minutes period.

What is λ ?
$\lambda = 2$ per 5 minutes.
What kind of probability are we looking for?
$P(X>2|\lambda = 2)$.

$$\begin{aligned}
&P(X > 2|\lambda = 2) \\
&= 1 - P(X \leq 2|\lambda = 2) \\
&= 1 - [P(X = 0) + P(X = 1) \\
&\quad + P(X = 2)] \\
&\quad = 1 - [\frac{e^{-2}2^{0}}{0!} + \frac{e^{-2}2^{1}}{1!} \\
&\quad + \frac{e^{-2}2^{2}}{2!}] \\
&\quad = 1 - (e^{-2}(1 + 2 + 2)) \\
&\quad = 0.3233
\end{aligned}$$

Practice

$$P(X) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

- Customers arrive at a photocopy center at an average rate of two every 5 minutes.
- Find a probability that more than two customers arrive in a 1-minute period.

X: # of arriving customers in a 1 minutes period.

What is λ ?
λ =2/5.
What kind of probability are we looking for?
$P(X>2|\lambda = 2/5)$.

Practice

$$P(X) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

- A professor receives on average 4.2 e-mails from students the day before a final exam. If the distribution of receiving the e-mails is Poisson, what is the probability of receiving at least 3 of these e-mails on such a day?

What is λ? = 4.2
$P(X\geq 3|\lambda=4.2)=1-[P(X=2)+P(X=1)+P(X=0)]$
=1-e^-4.2[1+4.2+(4.2^2)/2] = 0.78976