Discrete Probability Distribution

Week 12

Joint Probability Function

- A joint probability function is used to express the probability that X takes the specific value x and simultaneously Y takes the value y, as a function of x and y.
- $P(x,y)=P(X=x \cap Y=y)$

Practice

- Coin toss – 3 times
- X: # of heads on the 1st toss
- Y: total number of heads
- S={hhh, hht, hth, htt, thh, tht, tth, ttt}
- Complete the table below, joint probabilities and marginal probabilities

		Y (total number of heads in 3 tosses)				P(X)
		0	1	2	3	
X (# of heads on 1st toss)	0					
	1					
P(Y)						

		Y (total number of heads in 3 tosses)				P(X)
		0	1	2	3	
X (# of heads on 1st toss)	0	{ttt} =1/8	{tht, tth} =2/8	{thh} =1/8	0	=4/8
	1	0	{htt} =1/8	{hht, hth} =2/8	{hhh} =1/8	=4/8
P(Y)		=1/8	=3/8	=3/8	=1/8	

Conditional Probability Function

- The conditional probability function of the random variable Y expresses the probability that Y takes the value y when the value x is specified for X.

$$P(y|x) = \frac{P(x,y)}{P(x)} \qquad P(x|y) = \frac{P(x,y)}{P(y)}$$

		Y (total number of heads in 3 tosses)				P(X)
		0	1	2	3	
X (# of heads on 1st toss)	0	{ttt} =1/8	{tht, tth} =2/8	{thh} =1/8	0	P(X=0)= 4/8
	1	0	{htt} =1/8	{hht, hth} =2/8	{hhh} =1/8	P(X=1)= 4/8
P(Y)		P(Y=0) =1/8	P(Y=1) =3/8	P(Y=2) =3/8	P(Y=3)= 1/8	

- Find P(X=1|Y=2) and P(Y=2|X=1)

$$P(x|y) = \frac{P(x,y)}{P(y)} \qquad P(y|x) = \frac{P(x,y)}{P(x)}$$

Independence

- The jointly distributed random variables X and Y are said to be independent iff their joint probability function is the product of their marginal probability functions.
- P(x,y) = P(x)P(y) for all possible pairs of values x and y.

$\Rightarrow$P(y|x) = [P(x)P(y)]/P(x) = P(y)
$\Rightarrow$P(x|y)=[P(x)P(y)]/P(y) = P(x)

		Y (total number of heads in 3 tosses)				P(X)
		0	1	2	3	
X (# of heads on 1st toss)	0	{ttt} =1/8	{tht, tth} =2/8	{thh} =1/8	0	=4/8
	1	0	{htt} =1/8	{hht, hth} =2/8	{hhh} =1/8	=4/8
P(Y)		=1/8	=3/8	=3/8	=1/8	

- Check if X and Y are independent.
- (Independence: P(x,y) = P(x)P(y) for all possible pairs of values x and y.)
- P(x=0,y=0)=?=P(x=0)P(y=0)
- 1/8 =?= (4/8)*(1/8) = 4/64=1/16   Not independent.
- (If you find one pair which is not satisfying the equality, you can state that they are not independent.)

Mean and Variance

- E(X)=X1*P(X1)+X2*P(X2)…+Xn*P(Xn)
- Var(X)=(X1-E(X))^2*P(X1)+(X2-E(X))^2*P(X2)+…
- (Xn-E(X))^2*P(Xn)
- Or Var(X)= X1^2*P(X1)+X2^2*P(X2)+…Xn^2*P(Xn)-E(X)^2

Covariance

- Let X and Y be discrete random variables with means μx and μy.

$$\mathrm{Cov}(X,Y) = E[(x-\mu_x)(y-\mu_y)] = \sum_x\sum_y (x-\mu_x)(y-\mu_y)P(x,y)$$

$$\mathrm{Cov}(x,y) = E[XY] - \mu_x\mu_y = \sum_x\sum_y xyP(x,y) - \mu_x\mu_y$$

Correlation

$$\rho = \mathrm{Cor}(x,y) = \frac{\mathrm{Cov}(x,y)}{\sigma_x\sigma_y}$$

- $-1 \le \rho \le 1$
- ρ = 0 : no linear relationship between x and y.
- ρ>0 : positive linear relationship between x and y
- ρ=1 : perfectly positive linear relationship
- ρ<0 : negative linear relationship between x and y

Linear Function of Several Random Variables

- W = aX+bY
- E(W)= $a\mu_X+b\mu_Y$

$$Var(W) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2abCov(X,Y)$$
$$Var(W) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2abCorr(X,Y)\sigma_X\sigma_Y$$

- W=aX-bY
- E(W)= $a\mu_X-b\mu_Y$

$$Var(W) = a^2\sigma_X^2 + b^2\sigma_Y^2 - 2abCov(X,Y)$$
$$Var(W) = a^2\sigma_X^2 + b^2\sigma_Y^2 - 2abCorr(X,Y)\sigma_X\sigma_Y$$

Practice

- Setting:
- New model of a cell phone
- Advertised on a TV program.

- 15% of people who watch the show regularly and could identify the product.
- 16% of people watch the show regularly
- 45% of people could identify the product.

- X=1: regularly watch the show, X=0 otherwise
- Y=1: correctly identify the product, Y=0 otherwise

functions

		Y(Identify Product)		
		0	1	P(X)
X(watch the show regularly)	0			
	1			
	P(Y)			1

- -15% of people who watch the show regularly and could identify the product.
- -16% of people watch the show regularly
- -45% of people could identify the product.

	0	1	P(X)
0	0.54	0.30	0.84
1	0.01	0.15	0.16
P(Y)	0.55	0.45	1

- e.g. P(0,0)=P(X=0 ∩ Y=0)=0.54

Practice

a. Find the joint probability function of X and Y.
b. Find conditional probability function of Y given X=1.
c. Are X and Y independent?
d. Find Cov(X,Y) and Cor(X,Y)
e. If W = 2X + Y, Find E(W) and Var(W)
f. If W = 2X – Y, Find E(W) and Var(W)

- b. Find conditional probability function of Y given X=1.

- P(Y=0|X=1)=P(X=1∩Y=0)/P(X=1) = 0.01/0.16=0.0625

- P(Y=1|X=1)=P(X=1∩Y=1)/P(X=1)=0.15/0.16
- =0.9375

c. Are X and Y independent?

- If independent, the following should be true:

- P(X,Y)=P(X|Y)P(Y)=P(X)P(Y)

- for all combinations of X and Y.

- For example, P(0,0)=P(X=0|Y=0)P(Y=0)=P(X=0)P(Y=0) ???

- P(0,0)=0.54 from the table.
- P(X=0)P(Y=0)=0.84*0.55=0.462. Since they are not equal, X and Y are NOT independent.

c. Find COV(X,Y) and CORR(X,Y)

- E(X)=0*0.84+1*0.16=0.16
- E(Y)=0*0.55+1*0.45=0.45
- Var(X)=(0-0.16)^2*0.84+(1-0.16)^2*0.16=0.1344
- Var(Y)=(0-0.45)^2*0.55+(1-0.45)^2*0.45=0.2475

- COV(X,Y)=0*0*0.54+0*1*0.30+1*0*0.01+1*1*0.15-0.16*0.45=0.078

- Cor(X,Y)=0.078/sqrt(0.1344)*sqrt(0.2475)=0.4277

$$\rho = \mathrm{Cor}(x,y) = \frac{\mathrm{Cov}(x,y)}{\sigma_x \sigma_y} \quad \mathrm{Cov}(x,y) = E[XY] - \mu_x\mu_y = \sum_x \sum_y xyP(x,y) - \mu_x\mu_y$$

e. If W = 2X + Y, Find E(W) and Var(W)

- E(W)=2*E(X)+E(Y)
- =2*0.16+1*0.45=0.77

- Var(W)=(2^2)*Var(X)+(1^2)*Var(Y)+2*2*1*Cov(X,Y)
- =4*0.1344+1*0.2475+4*0.078=1.0971

$$Var(W) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2abCov(X,Y)$$

f. If W = 2X - Y, Find E(W) and Var(W)

- E(X)=2*0.16-1*0.45=-0.13
- Var(X)=4*0.1344+1*0.2475-4*0.078=0.4731

$$Var(W) = a^2\sigma_X^2 + b^2\sigma_Y^2 - 2abCov(X,Y)$$

Continuous Probability Distribution

Week 12

Continuous Distributions covered.

- Uniform
- Normal
- Exponential

Continuous Random Variable

- A variable that can assume any value in an interval

e.g.

Time required to complete a task

Temperature of a solution

Thickness, height, weight

Probability Density Function (PDF): f(x)

- f(x) > 0 for all values of x.
- Area under f(x) over all values of x = 1.
- P(a<x<b) = area under f(x) between a and b

$$= \int_a^b f(x)dx$$

Cumulative Distribution Function (CDF): F(x)

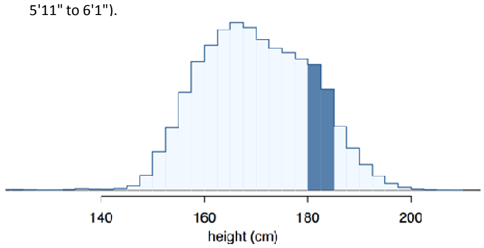
$$F(x) = P(X \leq x)$$

$$P(a < X < b) = F(b) - F(a)$$

$$F(x_0) = \int_{x_{min}}^{x_0} f(x)dx$$

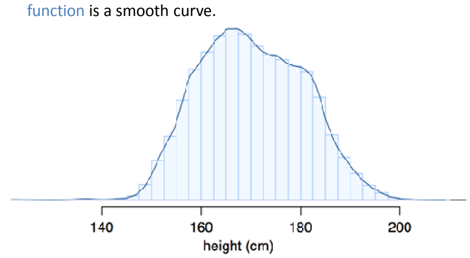
Continuous distributions

- Below is a histogram of the distribution of heights of US adults.
- The proportion of data that falls in the shaded bins gives the probability that a randomly sampled US adult is between 180 cm and 185 cm (about 5'11" to 6'1").



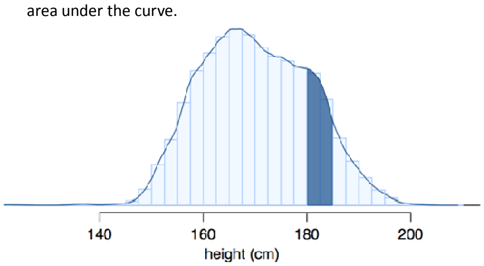
From histograms to continuous distributions

Since height is a continuous numerical variable, its probability density function is a smooth curve.



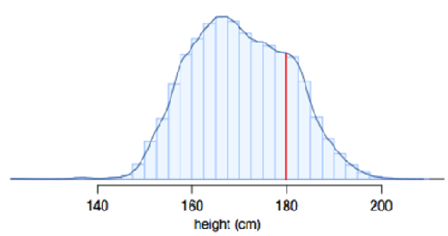
Probabilities from continuous distributions

Therefore, the probability that a randomly sampled US adult is between 180 cm and 185 cm can also be estimated as the shaded area under the curve.



By definition...

Since continuous probabilities are estimated as "the area under the curve", the probability of a person being exactly 180 cm (or any exact value) is defined as 0.



Quiz 9

- Q1. A company receives a shipment of 15 items. A random sample of 5 items is selected and the shipment is rejected if more than 1 defective item is found. What is the probability of accepting the shipment containing 4 defective items?
- Q2. Customers arrive at a post office at an average rate of 5 every 10 minutes. Find a probability that more than 1 customers arrive in a 5 minutes period.

Quiz 9_2

- Q1. A company receives a shipment of 20 items. A random sample of 5 items is selected and the shipment is rejected if more than 1 defective item is found. What is the probability of rejecting the shipment containing 6 defective items?
- Q2. Customers arrive at a post office at an average rate of 2 every 5 minutes. Find a probability that more than 2 customers arrive in a 10 minutes period.