Continuous Distribution

Week 14

Finding the value of x for a known probability

Step1: Find z value for known probability from the table.

Step2: convert z to x units using

$$x = \mu + z\sigma$$

Practice

- X~N(8,5)
- Find the value of x so that only 20% of all values are below this x value.

⇒Since z0 value for P(z<z0)=0.2 is not on the table, find instead P(z<z0')=0.8. z0' = -(z0). [will be illustrated on board].
⇒Find the nearest value F(z)=0.7995 for z=0.84, F(z)=0.8023 for z = 0.85. Since F(z)=0.7995 is closer to F(z)=0.8, choose z = 0.84.
⇒Since z0= -(z0')=-0.84, x = 8+(-0.84)*5=3.8.

Practice

- X~N(60, 15)
- Find the cutoff point for the top 10% of all observation.

⇒P(z<z0)=0.9. Look for F(z) value which is the closest to 0.9. z0=1.28.
⇒x=60+1.28*15=79.2.

Finding cutoff points

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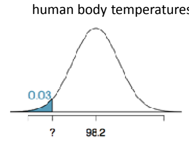

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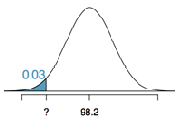

P(X<x)=P(Z<z0)=0.03. P(Z>-z0)=0.03.
P(Z<z0)=0.97. Look for F(z) = 0.97 or closest.
-z0=1.88. z0=-1.88. X=98.2-1.88*0.73=
96.8276

F=>C conversion (extra content)
= (96.8276-32)/1.8= 36.01.

$$P(X < x) = 0.03 \rightarrow P(Z < -1.88) = 0.03$$

$$Z = \frac{obs - mean}{SD} \rightarrow \frac{x - 98.2}{0.73} = -1.88$$

Practice

Body temperatures of healthy humans are distributed nearly normally with mean 98.2ºF and standard deviation 0.73ºF. What is the cutoff for the highest 10% of human body temperatures?

A. 97.3ºF
B. 99.1ºF
C. 99.4ºF
D. 99.6ºF

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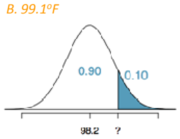

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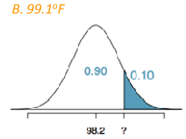

Z	0.05	0.06	0.07	0.08	0.09
1.0	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9115	0.9131	0.9147	0.9162	0.9177

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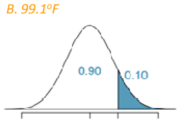

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$$P(X > x) = 0.10 \rightarrow P(Z < 1.28) = 0.90$$

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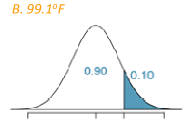

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$$P(X > x) = 0.10 \rightarrow P(Z < 1.28) = 0.90$$

$$Z = \frac{obs - mean}{SD} \rightarrow \frac{x - 98.2}{0.73} = 1.28$$

Practice

Body temperatures of healthy humans are distributed nearly normally with mean 98.2ºF and standard deviation 0.73ºF. What is the cutoff for the highest 10% of human body temperatures?

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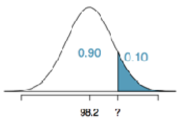

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$$P(X > x) = 0.10 \rightarrow P(Z < 1.28) = 0.90$$

$$Z = \frac{obs - mean}{SD} \rightarrow \frac{x - 98.2}{0.73} = 1.28$$

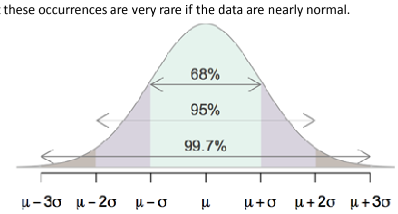
$$x = (1.28 \times 0.73) + 98.2 = 99.1$$

68-95-99.7 Rule

For nearly normally distributed data,

- about 68% falls within 1 SD of the mean,
- about 95% falls within 2 SD of the mean,
- about 99.7% falls within 3 SD of the mean.

It is possible for observations to fall 4, 5, or more standard deviations away from the mean, but these occurrences are very rare if the data are nearly normal.



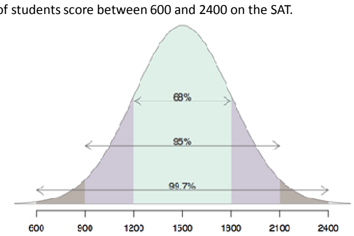
Describing variability using the 68-95-99.7 Rule

SAT scores are distributed nearly normally with mean 1500 and standard deviation 300.

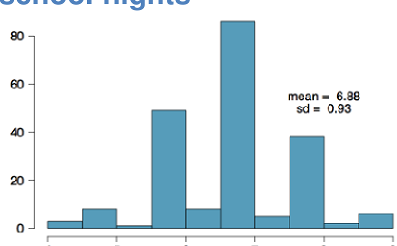
Describing variability using the 68-95-99.7 Rule

SAT scores are distributed nearly normally with mean 1500 and standard deviation 300.

- ~68% of students score between 1200 and 1800 on the SAT.
- ~95% of students score between 900 and 2100 on the SAT.
- ~99.7% of students score between 600 and 2400 on the SAT.

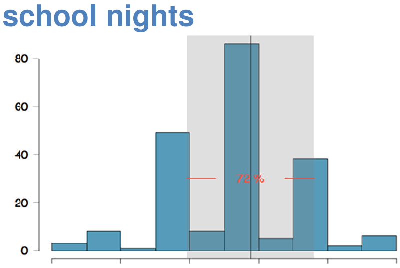


Number of hours of sleep on school nights



- Mean = 6.88 hours, SD = 0.92 hrs

Number of hours of sleep on school nights



- Mean = 6.88 hours, SD = 0.92 hrs
- 72% of the data are within 1 SD of the mean: 6.88 ± 0.93

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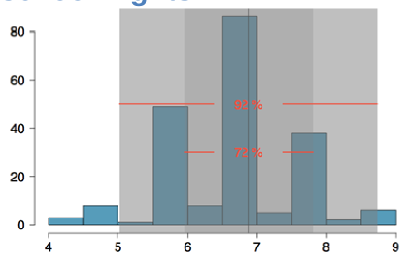

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- 92% of the data are within 1 SD of the mean: 6.88 ± 2 x 0.93

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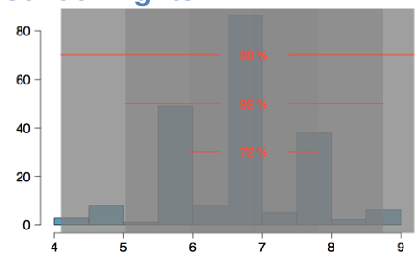

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- 72% of the data are within 1 SD of the mean: 6.88 ± 0.93
- 92% of the data are within 1 SD of the mean: 6.88 ± 2 x 0.93
- 99% of the data are within 1 SD of the mean: 6.88 ± 3 x 0.93

Practice

Which of the following is false?

A. Majority of Z scores in a right skewed distribution are negative.
B. In skewed distributions the Z score of the mean might be different than 0.
C. For a normal distribution, IQR is less than 2 x SD.
D. Z scores are helpful for determining how unusual a data point is compared to the rest of the data in the distribution.

Practice

Which of the following is false?

A. Majority of Z scores in a right skewed distribution are negative.
B. *In skewed distributions the Z score of the mean might be different than 0.*
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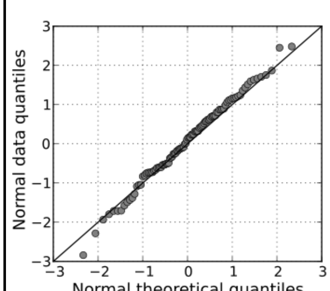
D. Z scores are helpful for determining how unusual a data point is compared to the rest of the data in the distribution.

Evaluating the normal approximation

Q-Q (Quartile-Quartile) plot



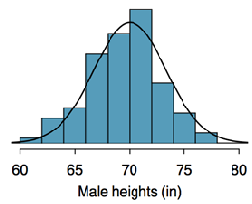
- The linearity of the points suggests that the data are normally distributed.

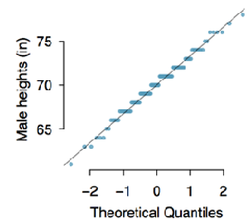
Q-Q Plot (Quartile-Quartile Plot)

- In R, **qqnorm( )** function can be used to create a Quantile-Quantile plot evaluating the fit of sample data to the normal distribution.
- More generally, the **qqplot( )** function creates a Quantile-Quantile plot for any theoretical distribution to test if two data sets come from populations with a common distribution.

Normal probability plot

A histogram and *normal probability plot* of a sample of 100 male heights.



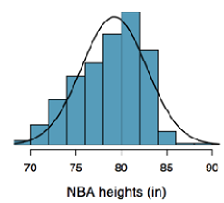


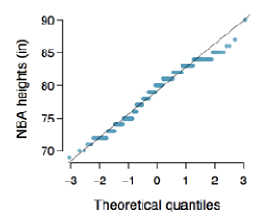
Anatomy of a normal probability plot

- Data are plotted on the y-axis of a normal probability plot, and theoretical quantiles (following a normal distribution) on the x-axis.
- If there is a linear relationship in the plot, then the data follow a nearly normal distribution.
- Constructing a normal probability plot requires calculating percentiles and corresponding z-scores for each observation, which is tedious. Therefore we generally rely on software when making these plots.

Practice

Below is a histogram and normal probability plot for the NBA heights from the 2008-2009 season. Do these data appear to follow a normal distribution?





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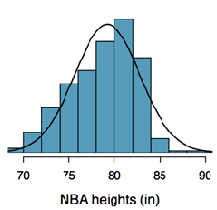
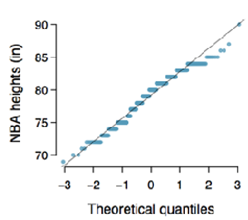
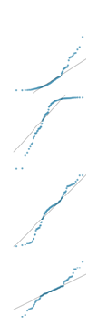




Why do the points on the normal probability have jumps?

Normal probability plot and skewness



Right skew - Points bend up and to the left of the line.

Left skew - Points bend down and to the right of the line.

Short tails (narrower than the normal distribution) - Points follow an S shaped-curve.

Long tails (wider than the normal distribution) - Points start below the line, bend to follow it, and end above it.

Normal Distribution Approximation for Binomial Distribution

- Used to compute probabilities for large sample sizes when Binomial tables are not available.
- Binomial Distribution $P(x) = \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$

  where $E(x) = np, Var(x) = np(1-p)$

- Good approximation if np(1-p)> 9

$$Z = \frac{x - E(x)}{\sqrt{Var(x)}} = \frac{x - np}{\sqrt{np(1-p)}}$$

- X~ Bin(n,p). However, if np(1-p)>9, then treat as X~ N(np, sqrt[np(1-p)]).

$$Z = \frac{x - E(x)}{\sqrt{Var(x)}} = \frac{x - np}{\sqrt{np(1-p)}}$$

- Probabilities can be solved as

$$P(a \le x \le b) = p\left(\frac{a - np}{\sqrt{np(1-p)}} \le \frac{x - np}{\sqrt{np(1-p)}} \le \frac{b - np}{\sqrt{np(1-p)}}\right)$$

Practice

- A salesman makes initial phone contact, then visit their homes if she assesses it is worthwhile. She knows that 40% of phone calls lead to follow-up visits.

Q: If she contact 100 people by phone, what is the probability that between 45 and 50 home visits will result?

Practice

- A salesman makes initial phone contact, then visit their homes if she assesses it is worthwhile. She knows that 40% of phone calls lead to follow-up visits.

Q: If she contact 100 people by phone, what is the probability that between 45 and 50 home visits will result?

1: Direct method (use Binomial Distribution)

$$P(45 \le x \le 50) = P(x = 45) + P(x = 46) + \cdots + P(x = 50)$$
$$= \frac{100!}{45!\,55!} 0.4^{45} 0.6^{55} + \frac{100!}{46!\,54!} 0.4^{46} 0.6^{54} \ldots$$
$$+ \frac{100!}{50!\,50!} 0.4^{50} 0.6^{50} = ???$$

Or by using Binomial Table:
F(50|n=100,p=0.4)-F(45|n=100,p=0.4) = ?

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Q: If she contact 100 people by phone, what is the probability that between 45 and 50 home visits will result?

2: Using Normal Approximation [np(1-p)=100*0.4*0.6=24 > 9]

E(x)=np=100*0.4 = 40, Var(x)=np(1-p)=100*0.4*0.6=24.

$$P(45 \le x \le 50) = P\left(\frac{45 - 40}{\sqrt{24}} \le z \le \frac{50 - 40}{\sqrt{24}}\right)$$
$$= P(1.02 \le z \le 2.04) = F(2.04) - F(1.02) =$$

Exponential Distribution

- Used for modeling waiting time or queuing problems.
- Positive Random Variable: t

$\Rightarrow$t: number of time units until next occurrence.

- Depends on a single parameter, $\lambda > 0$

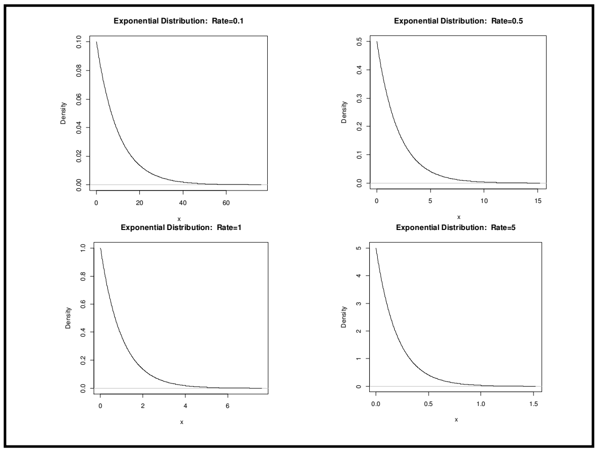
$\Rightarrow\lambda$: mean number of occurrences per time unit.

- Distribution is not symmetric

PDF: $f(t) = \lambda e^{-\lambda t}$ for $t \ge 0$

CDF: $F(t) = 1 - e^{-\lambda t}$ for $t \ge 0$

* Use CDF for the calculation of Probabilities!



Comparing with Poisson Distribution

- Poisson: probability of **x successes** during a time unit (e.g. 5 minutes): RV (X) is # of successes (discrete variable).

$$P(X) = \frac{e^{-\lambda}\lambda^x}{x!}$$

- Exponential: probability that **a success** will occur during **an interval of time t**. RV (t) is time (continuous variable).

$$f(t) = \lambda e^{-\lambda t} \text{ for } t \geq 0$$

Practice $F(t) = 1 - e^{-\lambda t} \text{ for } t \geq 0$

- Service time at a repair shop can be modeled by an exponential distribution with mean service time of 5 minutes. What is the probability that a customer service time will take longer than 10 minutes?

- t: service time in minute
- λ: 1/5 service per minute (<= 1 service per 5 min.)

- Service time at a repair shop can be modeled by an exponential distribution with mean service time of 5 minutes. What is the probability that a customer service time will take longer than 10 minutes?

$$F(t) = 1 - e^{-\lambda t} \text{ for } t \geq 0$$

- t: service time in minute
- λ: 1/5 service per minute (<= 1 service per 5 min.)

P(t>10)=1-F(10)=1- [1- e^(-0.2*10)]=e^(-2)=0.1353.

The probability that the service time exceeds 10 minutes is 0.1353 or 13.53%.

Practice

- A professor sees students during his office hour. Time spent with students follow an exponential distribution with mean 10 minutes.

Q1: Find the probability that a given student spends less than 20 mins with the professor.

Q2: Find the probability that a given student spends more than 5 mins with the professor.

- A professor sees students during his office hour. Time spent with students follow an exponential distribution with mean 10 minutes.

Q1: Find the probability that a given student spends less than 20 mins with the professor.

Q2: Find the probability that a given student spends more than 5 mins with the professor.

λ = 1/10 meeting per minute.
(<=1 meeting per 10 mins)

Q1: P(t<20)=1-e^(-0.1*20)=1-e^-2=0.865

Q2: P(t>5)=1-[1-e^(-0.1*5)]=e^-0.5=0.607

Jointly Distributed Continuous R.V.

- Joint Cumulative Distribution Function

Let X1 and X2 be continuous R.V.

1. Joint CDF: F(X1, X2) = P(X1<x1∩X2<x2)
2. Marginal distribution function : F(X1)=P(X1<x1)
3. RVs are independent iff F(X1,X2)=F(x1)F(x2)
4. Cov(X,Y)=E[(X-μx)(Y-μy)] = E(XY)-μxμy.
5.

$$\mathrm{Cor}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$W = aX + bY$$

$$E(W) = a\mu_X + b\mu_Y$$

$$\begin{aligned}\mathrm{Var}(W) &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Cov}(X, Y) \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 \\ &\quad + 2ab\mathrm{Cor}(X, Y)\sigma_X\sigma_Y\end{aligned}$$

Practice

Px~N(25, 9), Py~N(40,11)

- Nx=20 (# of Stock X)
- NY=30 (# of Stock Y)
- Cor(X,Y)=-0.40.

Q: find the probability that the portfolio (W=20Px+30Py) value exceeds 2000.

Practice

Px~N(25, 9), Py~N(40,11)

- Nx=20 (# of Stock X)
- NY=30 (# of Stock Y)
- Cor(X,Y)=-0.40.

Q: find the probability that the portfolio (W=20Px+30Py) value exceeds 2000.

E(W)=20*25+30*40=1700
Var(W)=400*81+900*121+2*20*30*(-0.4)*9*11 = 98780
Stdev(W)=306.24

P(W>2000)=P(z> (2000-1700)/306.24) = P(z>0.98)=1-F(0.98)=1-0.8365=0.1635.
The probability for the portfolio value to exceed 2000 is 16.35%.

Bonus Quiz

X~N(56, 11)

Q1: Find P(42<X<75).

Q2: Find the value of X so that 10% of all values are below this value.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817

Bonus Quiz _2

- X~ N(68, 15)

Q1: Find P(55<X<80)

Q2: Find the value of X so that only 10% of all values are below this X.