ECO106/138 Week 1 Lecture Note Template

NOTE: If you forgot the basics of integral, review the lecture note/video from ECO105/137 Week14 on www.shihomiaksoy.org

+10

$$\int_a^b f(x)\,dx = \boxed{} + C$$

Definite Integrals

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Properties

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii) $\int_a^a f(x)\,dx = 0$

(iii) $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(v) $\int_a^b [\alpha f(x) + \beta g(x)]\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$

e.g. 1

$$\int_0^5 (x^1 + x^2)\,dx = \left[\frac{1}{1+1}x^{1+1} + \frac{1}{2+1}x^{2+1}\right]\Big|_0^5$$

$$= \left(\frac{1}{2}(5)^2 + \frac{1}{3}(5)^3\right) - \left(\frac{1}{2}(0)^2 + \frac{1}{3}(0^3)\right)$$

$$= \frac{75+250}{6} = \frac{325}{6} = 54.13$$

e.g.2

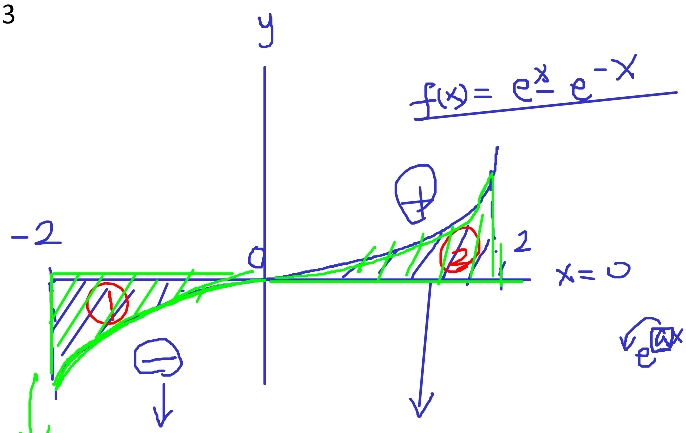
$$\int_1^2 (x^5 + x^{-5})\,dx$$

$$= \frac{1}{6}x^6 + \frac{1}{-5+1}x^{-5+1}\Big|_1^2$$

$$= \left[\frac{1}{6}(2)^6 - \frac{1}{4}(2)^{-4}\right] - \left[\frac{1}{6} - \frac{1}{4}\right]$$

$$= 10.73$$

e.g. 3



$$-\int_{-2}^{0}(e^x - e^{-x})\,dx + \int_0^2 (e^x - e^{-x})\,dx$$

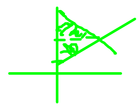
$$= -\left[e^x + e^{-x}\Big|_{-2}^{0}\right] + \left[e^x+e^{-x}\Big|_0^2\right]$$

$$= -[(1+1)-(e^{-2}+e^{2})] + [(e^{2}+e^{-2})-2]$$

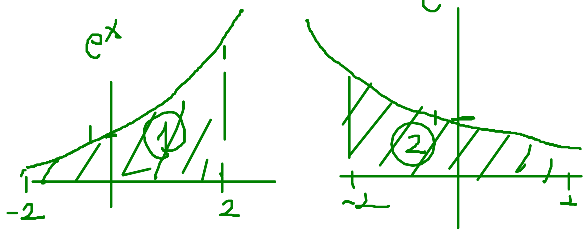
$$= -2+e^{-2}+e^{2}+e^{2}+e^{-2}-2$$

$$= 2e^{2}+2e^{-2}-4 = 11.05$$

$$\int e^{-x}\,dx = \int -e^{-x}\,dx$$



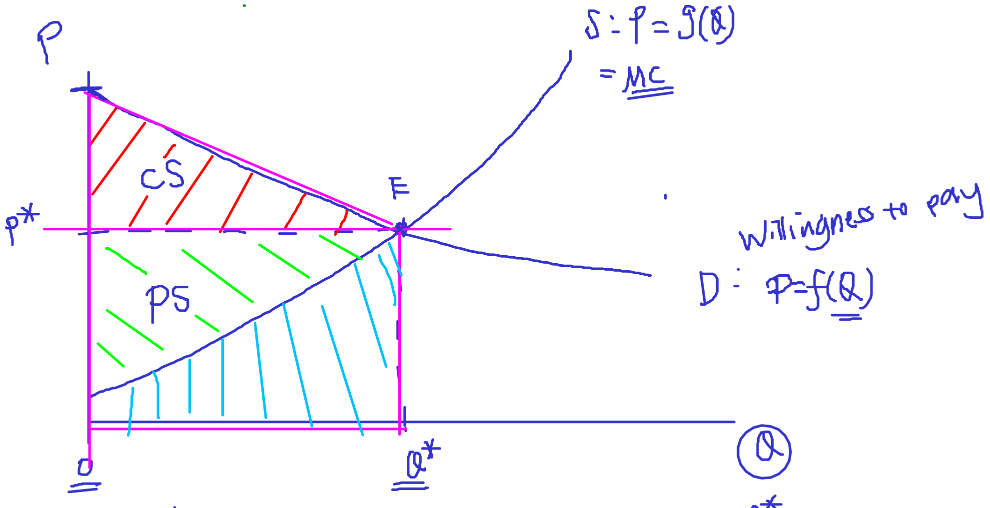
$$\int_{-2}^{2}(e^x - e^{-x})\,dx = 0$$



Economics Applications

Consumer Surplus and Producer Surplus

Illustration



$$CS = \int_0^{Q^*}[f(Q) - p^*]\,dQ$$

$$TS = \int_0^{Q^*}[f(Q) - g(Q)]\,dQ$$

$$PS = \int_0^{Q^*}[p^* - g(Q)]\,dQ$$

e.g. Linear Demand and Supply Functions

$D: \quad P = f(Q) = 50 - 0.1Q$

$S: \quad P = g(Q) = 0.2Q + 20$

Q. Find CS & PS.

① $Q^* = 100$, $P^* = 40$ $\Leftarrow$ $D = S$

$50 - 0.1Q = 0.2Q + 20$

$0.3Q = 30$

$Q^* = 100$

$P = 50 - 0.1(100) = 40.$

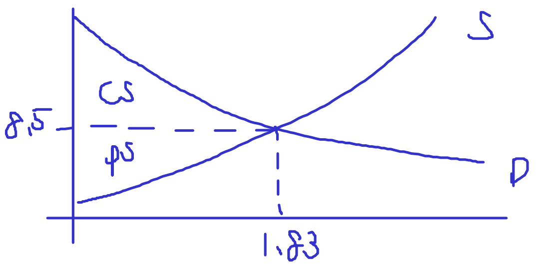
$$CS = \int_0^{100} (50 - 0.1Q - 40)\,dQ$$
$$= \int_0^{100} (10 - 0.1Q)\,dQ$$
$$= 10Q - 0.05Q^2 \Big|_0^{100}$$
$$= 10(100) - 0.05(100)^2 - (0-0)$$
$$= 1000 - 500 = 500$$

$$PS = \int_0^{100} (40 - (0.2Q + 20))$$
$$= \int_0^{100} (20 - 0.2Q)\,dQ$$
$$= 20Q - 0.1Q^2 \Big|_0^{100}$$
$$= 20(100) - 0.1(100^2) - (0-0)$$
$$= 1000 \quad //.$$

e.g. Quadratic Functions

$D: \quad P = \frac{1}{10}Q^2 - Q + 10$

$S: \quad P = \frac{1}{2}Q^2 + Q + 5$



① Find P^*, Q^*.

$$\frac{1}{10}Q^2 - Q + 10 = \frac{1}{2}Q^2 + Q + 5$$
$$-\frac{4}{10}Q^2 - 2Q + 5 = 0$$

QF: $$Q = \frac{2 \pm \sqrt{4 - 4(-\frac{4}{10})5}}{-\frac{8}{10}} = \frac{2 \pm \sqrt{12}}{-4/5} = \cancel{-6.83},\ \boxed{1.83}$$

$Q \geq 0$

$$P^* = \frac{1}{10}(1.83)^2 - 1.83 + 10 = \boxed{8.5}$$

$$CS = \int_0^{\boxed{1.83}} \left(\frac{1}{10}Q^2 - Q + 10 - \underset{P^*}{8.5}\right)dQ = \frac{1}{30}Q^3 - \frac{1}{2}Q^2 + 1.5Q \Big|_0^{1.83}$$

(Q^*)

$$= \frac{1}{30}(1.83)^3 - \frac{1}{2}(1.83)^2 + 1.5(1.83) = 1.275$$

$$PS = \int_0^{\boxed{1.83}} \left(\underset{P^*}{8.5} - \left(\frac{1}{2}Q^2 + Q + 5\right)\right)dQ$$

$$= 3.5Q - \frac{1}{2}\cdot\frac{1}{3}Q^3 - \frac{1}{2}Q^2 \Big|_0^{1.83} = 3.5(1.83) - \frac{1}{6}(1.83)^3 - \frac{1}{2}(1.83)^2 = 3.71 \quad //$$

Integration by Parts: For Indefinite Integral

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

Proof: $\quad (f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$\underbrace{\int (f(x)g(x))'}_{f(x)g(x)} = \int f'(x)g(x)\,dx + \underbrace{\int f(x)g'(x)\,dx}_{LHS}$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

e.g. Find $\boxed{\int xe^x dx}$ by using integration by parts.

$$\begin{cases} f(x) = x & \to f'(x) = 1 \\ g'(x) = e^x & \to g(x) = e^x \end{cases} \qquad \int e^x dx$$

$$f(x) = e^x \to f'(x) = e^x, \quad g'(x) = x \to g(x) = \tfrac{1}{2}x^2 \qquad \int e^x \tfrac{1}{2}x^2\,dx$$

$$\int xe^x dx = xe^x - \int e^x dx$$
$$= xe^x - e^x + C \quad //$$

e.g. Find $I = \int \frac{1}{x} lnx dx$ by using integration by parts.

$$\begin{cases} f(x) = \ln x & \to f'(x) = \frac{1}{x} \\ g'(x) = \frac{1}{x} & \to g(x) = \ln x \end{cases} \qquad \int \frac{1}{x}\,dx$$

$$\underbrace{\int \frac{1}{x}\ln x\,dx}_{I} = (\ln x)^2 - \underbrace{\int \frac{1}{x}\ln x\,dx}_{I}$$

$$2I = (\ln x)^2 + C$$
$$I = \frac{(\ln x)^2}{2} + C \quad //$$

Integration by Parts: For Definite Integral

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

e.g. Solve $I = \int_{\boxed{0}}^{\boxed{10}} (1+0.4t)e^{-0.05t}dt$ by using integration by parts.

$\checkmark$ $\begin{cases} f(t) = 1+0.4t \to f'(t) = 0.4 \\ g'(t) = e^{-0.05t} \to g(t) = -20e^{-0.05t} \end{cases}$ $f'(x)g(x) = \boxed{-8e^{-0.05t}}$

$\begin{cases} f(t) = e^{-0.05t} \to f'(t) = (-0.05)e^{-0.05t} \\ g'(t) \quad 1+0.4t \to g(t) = t + \frac{0.4}{2}t^2 \end{cases}$ $f'(t)g(t) = (-0.05)e^{-0.05t}\left(t+\frac{0.4}{2}t^2\right)$

$\int(1+0.4t)\,dt$

$I = (1+0.4t)(-20e^{-0.05t})\Big|_0^{10} - \int_0^{10} -8e^{-0.05t}\,dt$

$= (1+0.4(10))(-20e^{-0.5}) - (1)(-20) + 8\int_0^{10} e^{-0.05t}\,dt$

$+ 8\left[-\frac{1}{0.05}e^{-0.05t}\right]\Big|_0^{10}$

$+ 8[-20(e^{-0.5}-1)]$

$= -100e^{-0.5} + 20 - 160e^{-0.5} + 160$

$= 180 - 260e^{-0.5} \approx 22.3$

e.g. Solve $\int_{-1}^{1} x\ln(x+2)dx$ by using integration by parts.

$\int \ln x\,dx = (x\ln x - x)$

$\begin{cases} f(x) = x \Rightarrow f'(x) = 1 \\ g'(x) = \ln(x+2) \Rightarrow g(x) = (x+2)\ln(x+2) - (x+2) \end{cases}$

$\checkmark$ $\begin{cases} f(x) = \ln(x+2) \Rightarrow f'(x) = \frac{1}{x+2} \\ g'(x) = x \Rightarrow g(x) = \frac{1}{2}x^2 \end{cases}$ $f'(x)g(x) = \frac{x^2}{2(x+2)}$

$\int x\,dx$

$\frac{x^2}{2(x+2)} = \frac{\overset{x^2-4}{(x+2)(x-2)+4}}{2(x+2)} = \frac{x-2}{2} + \frac{2}{x+2}$

$I = \left[\ln(x+2)\left(\frac{1}{2}x^2\right)\right]\Big|_{-1}^{1} - \int_{-1}^{1} \boxed{\frac{x^2}{2(x+2)}}dx$

$= \ln 3\left(\frac{1}{2}\right) - (\ln 1)\left(\frac{1}{2}\right)$

$= \frac{\ln 3}{2} - \frac{1}{2}\int_{-1}^{1}\left((x-2) + \frac{4}{x+2}\right)dx$

$= \frac{\ln 3}{2} - \frac{1}{2}\left[\left(\frac{1}{2}x^2 - 2x\right) + 4\ln(x+2)\right]\Big|_{-1}^{1} = \frac{\ln 3}{2} - \frac{1}{2}\left(\left(\frac{1}{2} - 2 + 4\ln 3\right) - \left(\frac{1}{2} + 2 + 4\ln 1\right)\right)$

$= \frac{1}{2}\ln 3 + 2 - 2\ln 3$

$= 2 - \frac{3}{2}\ln 3 = 0.352$