ECO106/138 Week 9 Lecture Note Template

Chain Rule for Many Variables

$$If\ z = F(x, y), with\ x = f(t, s) and\ y = g(t, s)$$

Then (a) $\frac{\partial z}{\partial t} =$

(b) $\frac{\partial z}{\partial s} =$

e.g. $z = x^2 + 2y^2, x = t - s^2, y = ts. Find\ \frac{\partial z}{\partial t}, \frac{\partial z}{\partial s}.$

$x(t,s)$, $y(t,s)$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 2x(1) + 4y(s) = 2(t-s^2) + 4(ts)(s)$$
$$= 2t - 2s^2 + 4ts^2$$

$$\frac{\partial z}{\partial s} = -4xs + 4ty = -4(t-s^2)\cdot s + 4t(ts)$$
$$= -4ts + 4s^3 + 4t^2 s$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left[2x e^{x^2} + y^3 e^{xy}\right](2) + \left[2y e^{xy} + x y^2 e^{xy}\right](2ts^3) \rightarrow$$

e.g. $If\ z = e^{x^2} + y^2 e^{xy}, x = 2t + 3s, y = t^2 s^3, find\ z_t'(t = 1, s = 0).$

$x = 2$, $y = 0$

$$\rightarrow = 2\cdot 2\cdot 2e^4 = 8e^4 = 437$$

$z = F(x_1, x_2)$, $x_1 = f_1(t,s)$, $x_2 = f_2(t,s)$

General Chain Rule

If $z = F(x_1, \cdots x_n)$ with $x_1 = f_1(t_1 \cdots t_j \cdots t_m)$

GDP $\qquad x_2 = f_2(t_1 \cdots t_j \cdots t_m)$

$x_n = f_n(t_1 \cdots t_j \cdots t_m)$

$$\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_j} \qquad j = 1 \cdots m.$$

marginal effect of t_j on z

If $t_j \uparrow$ by 1 unit, $\rightarrow \Delta z$?

e.g. $Y = F(K, L, T), K = K(t), L = L(t), T = T(t). Find\ \frac{\partial Y}{\partial t}.$

$\leftarrow 1\% \uparrow K \rightarrow a\%\ Y$

If $Y = F(K, L, T) = AK^a L^b T^c$, derive relative rate of change of output. $\frac{\dot{Y}}{Y} = \frac{\partial Y/\partial t}{Y}$

$$\dot{Y} = \frac{\partial Y}{\partial t} = \left[\frac{\partial F}{\partial K}\frac{\partial K}{\partial t}\right] + \frac{\partial F}{\partial L}\cdot\frac{\partial L}{\partial t} + \frac{\partial F}{\partial T}\cdot\frac{\partial T}{\partial t}$$

$$\frac{\dot{Y}}{Y} = \left(\left[aAK^{a-1}L^bT^c\right]\left(\frac{\partial K}{\partial t}\right) + \left[bAK^aL^{b-1}T^c\right]\left(\frac{\partial L}{\partial t}\right) + \left[cAK^aL^bT^{c-1}\right]\left(\frac{\partial T}{\partial t}\right)\right) \Big/ AK^aL^bT^c \leftarrow Y$$

$$= a\frac{\dot{K}}{K} + b\frac{\dot{L}}{L} + c\frac{\dot{T}}{T}$$

RRC

$z = s^2$

e.g. $w = x^2 + y^2 + z^2, x = \sqrt{t+s}, y = e^{ts}, Find\ \frac{\partial w}{\partial t}.$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= [2x]\left(\tfrac{1}{2}(t+s)^{-1/2}\right) + [2y]\left(se^{ts}\right) + [2z](0)$$

$$= \frac{2}{2}(t+s)^{1/2}(t+s)^{-1/2} + 2(e^{ts})(se^{ts})$$

$$= 1 + 2se^{2ts}$$

Implicit Differentiation

How do we find $\frac{\partial y}{\partial x}$ if a function is defined implicitly?

$F(x,y)$

e.g. $x^3 + x^2y - 2y^2 - 10y = 0. Find\ \frac{\partial y}{\partial x}$

$$F(x,y) = c \qquad \frac{\partial y}{\partial x} = -\frac{F'_x(x,y)}{F'_y(x,y)}$$

Derivation

$$\left.\begin{array}{l} y = f(x) \\ f(x,y) = c \end{array}\right\} \quad F(x, f(x)) = c$$

$$\frac{\partial F}{\partial x} = F'_x + F'_y \cdot \frac{\partial y}{\partial x} = 0$$

$$\boxed{\frac{\partial y}{\partial x} = -\frac{F'_x}{F'_y}}$$

$F(x,y)$

e.g. $x^3 + x^2y - 2y^2 - 10y = 0. Find\ \boxed{\frac{\partial y}{\partial x}}$

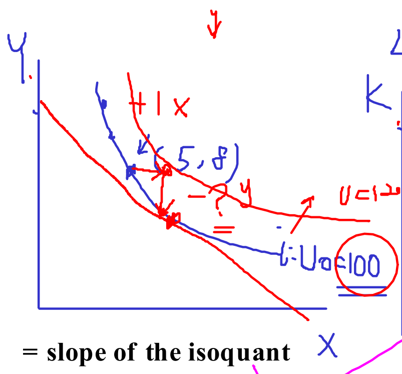
$$\frac{\partial y}{\partial x} = -\frac{3x^2 + 2xy}{x^2 - 4y - 10} \quad \begin{array}{l} \leftarrow F'_x \\ \leftarrow F'_y \end{array}$$

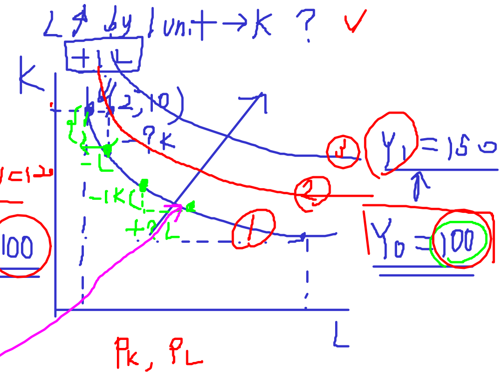
e.g.

$$\underline{Y_0 = 10K^{\boxed{1/2}}L^{\boxed{1/2}}}.\ Find\ \boxed{\left|\frac{\partial K}{\partial L}\right|}$$

marginal rate of technical substitution

$$\frac{\partial K}{\partial L} = -\frac{F'_L}{F'_K} = \frac{-5K^{1/2}L^{-1/2}}{5K^{-1/2}L^{1/2}} = -\frac{K}{L}$$



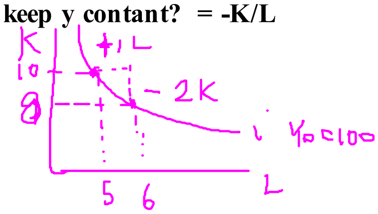


$$\Rightarrow \left|\frac{\partial K}{\partial L}\right| = \frac{K}{L}$$ = slope of the isoquant

If L is increasd by 1 unit, how much k has to be decreased to keep y contant? = -K/L

if $K = 10,\ L = 5$

$$\left|\frac{\partial K}{\partial L}\right| = \frac{10}{5} = 2$$



e.g.

$$Y = 10K^{1/4}L^{1/2}$$

$$\left|\frac{\partial K}{\partial L}\right| = \frac{5K^{1/4}L^{-1/2}}{\frac{5}{2}K^{-3/4}L^{1/2}} = 2\frac{K}{L} = 4$$

at $K = 10,\ L = 5$

e.g. $e^{xy^2} - 2x - 4y = c.$

1. $When\ (x, y) = (0,1), find\ c.$ $e^0 - 0 - 4 = -3$

2. $Find\ \dfrac{\partial y}{\partial x}\ and\ evaluate\ at\ (0,1)$

$$\frac{\partial y}{\partial x} = -\frac{F'_x}{F'_y} = -\frac{(y^2e^{xy^2} - 2)}{[2xye^{xy^2} - 4]} = -\frac{(1-2)}{-4} = -\frac{1}{4}$$

General Case

$$F(x,y,z)=c \Rightarrow z'_x = -\frac{F'_x}{F'_z}, \quad z'_y = -\frac{F'_y}{F'_z}$$

e.g. $x-2y-3z+z^2=-2. \quad z=f(x,y).$

Find $z''_{xx}, z''_{xy}, z''_{yy}$

and evaluate at $(x,y,z)=(0,0,2)$

$$F(x)=\frac{f(x)}{g(x)}$$

$$F'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$$

$f(x)=-1$

$g(x)=-3+2z$

$$\rightarrow z'_x = -\frac{F'_x}{F'_z} = -\left(\frac{1}{-3+2z}\right) = -1$$

$$\rightarrow z'_y = -\frac{F'_y}{F'_z} = -\left(\frac{-2}{-3+2z}\right) = 2$$

$$\rightarrow z''_{xx} = \frac{0-(-2)z'_x}{(-3+2z)^2} = \frac{2(-1)}{(-3+4)^2} = \boxed{-2}$$

$$\rightarrow z''_{xy} = \frac{-(-2)z'_y}{(-3+2z)^2} = \frac{2(2)}{(-3+4)^2} = \boxed{4}$$

$$\rightarrow z''_{yy} = \frac{-2(2z'_y)}{(-3+2z)^2} = \frac{-4(2)}{1} = \boxed{-8}$$

$$z'_x = -\frac{1}{(-3+2z)} \quad \partial y$$

$$z''_{xy} = \frac{0-(-1)2z'_y}{(-3+2z)^2} = \frac{2(2)}{1}$$

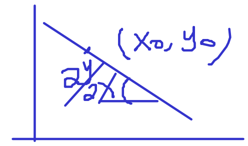
$$F(x_1,x_2 \ldots x_n, z) = c$$

$$\Rightarrow \frac{\partial z}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}, \quad i=1,2\ldots n$$

$$z=f(x_1,x_2\cdots x_n)$$

Point-slope formula $(y-y_0)=\frac{\partial y}{\partial x}(x-x_0)$

point-point $(y-y_0)=\frac{y_1-y_0}{x_1-x_0}(x-x_0)$



e.g. The function F is defined for all x and y by $F(x,y)=xe^{y-3}+xy^2-2y$. Find the equation for the tangent line to the curve at the point (1,3).

$(1,3)$

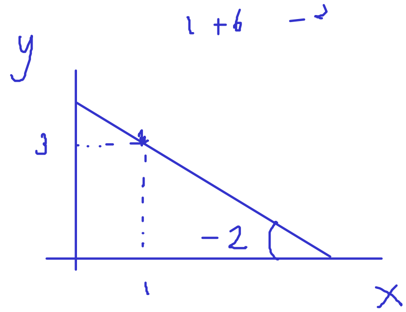
$$\frac{\partial y}{\partial x}=-\frac{F_x'}{F_y'}=-\frac{e^{y-3}+y^2}{xe^{y-3}+2xy-2}=-\frac{1+9}{1\cdot 1+2\cdot 1\cdot 3-2}=-\frac{10}{5}=-2.$$

$1+6 \quad -2$

$(y-3)=-2(x-1)$

$y=-2x+2+3$

$=-2x+5$ //



Homogeneous Function

A function f of n variables x1,… xn defined in a domain D is said to be homogeneous of degree K if for all (x1,…xn) in D,

$$f(tx_1\cdots tx_n)=\boxed{t^k}f(x_1,\ldots x_n) \qquad \text{HD } k$$

e.g. $f(x_1,x_2)=Ax_1^ax_2^b$, $with\ x_1\geq 0, x_2\geq 0.$ What's HD?

$f(tx_1,tx_2)=A(tx_1)^a(tx_2)^b$

$=t^{a+b}Ax_1^ax_2^b=t^{a+b}f(\ )$ HD $a+b$

e.g. $f(x, y) = 3x^2y - y^3$ What's HD?

$$f(tx, ty) = 3(tx)^2(ty) - (ty)^3$$
$$= t^3(3x^2y - y^3) \quad \text{HD } 3$$

e.g. $f(x_1, x_2) = x_1 + x_2^2$ What's HD?

$$f(tx_1, tx_2) = tx_1 + (tx_2)^2$$
$$= \underbrace{t}(x_1 + tx_2^2) \quad \times \quad \text{Not a homogeneous fn.}$$

$f(x_1, x_2) = Ax_1^a x_2^b$

if $a+b=1 \Rightarrow$ Constant returns to scale

$\rightarrow t *$ input $\Rightarrow t *$ output

e.g. $Ax_1^{1/2}x_2^{1/2}$

$Ax_1^{1/3}x_2^{2/3}$

$a+b>1 \rightarrow$ IRS

$a+b<1 \rightarrow$ DRS

Partial Derivative of Homogeneous Function

Let f be a differentiable function of n variables that is HD k.

Then each of its partial derivaties f_i' $(for\ i = 1 \dots n)$ is HD (k-1).

Proof: $f(tx_1, \ldots tx_i \ldots tx_n) = t^k f(x_1 \cdots x_n) \leftarrow \boxed{\text{HD } k}$

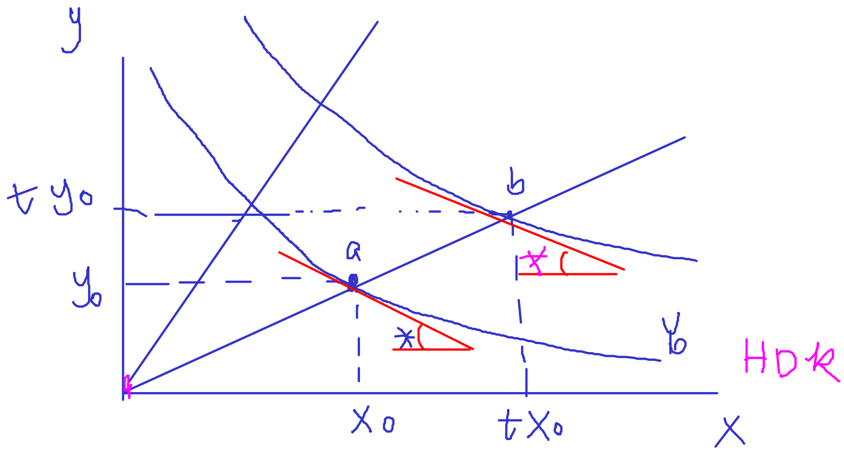
Differentiate both sides with respect to xi

$$t f'(tx_1 \cdots tx_i \cdots tx_n) = t^k f'(x_1 \cdots x_n)$$

$$f'(tx_1 \cdots tx_n) = t^{k-1} f'(x_1 \cdots x_n) \leftarrow \boxed{\text{HD } k-1}$$

⇨ Application

⇨ Slopes of level curves of Homogeneous Function.



Slope at a $\quad \dfrac{dy}{dx} = -\dfrac{F'_x(x,y)}{F'_y(x,y)}$ *

slope at b $\quad \dfrac{dy}{dx} = -\dfrac{F'_x(tx,ty)}{F'_y(tx,ty)} = -\dfrac{\cancel{t^{k-1}}F'_x(x,y)}{\cancel{t^{k-1}}F'_y(x,y)} = -\dfrac{F'_x(x,y)}{F'_y(x,y)}$ *

$F(tx,ty) = t^k F(x,y)$

F_x HD k $\qquad F'(tx,ty) = \boxed{t^{k-1}F'(x,y)}$

F'_x HD $k-1$

If F be a differentiable function of two varialbes that is HD k.
Then along any given ray from the origin, the slopes of the level curves of F are the same.