ECO106/138

Functions of Many Variables

Topics:
1. Functions of Two Variables
2. Functions of More Variables
3. Partial Elasticities
4. Level Curves

Before $y = f(x)$

1. Functions of Two variables

$z = f(x, y)$

dependent var — independent var.

e.g. $z = QD$ for organic egg ⇒ what are the determinates of QD?

$x =$ price

$y =$ income (HH)

$QD = f(\text{price}^{(-)}, \text{income}^{(+)})$

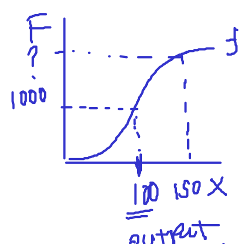
→ $QD = f$(price, income, pref, price(regular) ..)

e.g. $f(x,y) = 2x + x^2y^3$

$f(1,1) = 2(1) + (1)^2(1)^3 = 3$

$f(0,1) = 0$

$f(3,1) = 2\cdot 3 + 3^2\cdot 1^3 = 15$



e.g. Cobb-Douglas Production Function

$a+b=1 \Rightarrow$ CRS doubly input → doubly output

① $F(x,y) = Ax^a y^b$

$F = AK^\alpha L^\beta$

$a+b<1 \Rightarrow$ DRS double inputs ⇒ $t^{a+b} < 2$

$a+b>1 \Rightarrow$ IRS doubly inputs $t^{a+b} > 2$

x = amount of capital inputs (K)

y = amount of labor inputs (L)

elasticity

1% ↑ in X → a% ↑ in F

1% ↑ in Y → b% ↑ in F

F(x,y) = amount of production.

e.g. $F(tx, ty) = A(tx)^a(ty)^b = t^{a+b}Ax^ay^b$, $t=2$

$t=2$ if you double both inputs what's happen to F?

② $\varepsilon_x = \boxed{\frac{\partial F}{\partial x}} \cdot \frac{x}{F} = aAx^{a-1}y^b \cdot \frac{x}{Ax^ay^b} = a$

X ↑ 1 unit, F ↑ by $\frac{\partial F}{\partial x}$ unit

(a) double

(b) 4 times

(c) less than double

(d) more than 2 ($2<F<4$) b/w 2 & 4

(e) depends on the elasticity (a, b)

e.g. F(K,L) =$10K^{1/2}L^{1/2}$

since $a+b=1$

when K, L are doubled output → doubling output.

$F(100,25) = 10(100)^{1/2}(25)^{1/2} = 500$

Double inputs

$F(200,50) = 10(200)^{1/2}(50)^{1/2} = 1000$ ⇐ doubled.

$\varepsilon_y = \frac{\partial F}{\partial y}\cdot\frac{y}{F} = \frac{(bAx^ay^{b-1})\cdot y}{Ax^ay^b} = b$

1% ↑ input y, b% ↑ F output.

$t^{a+b} = 2^{a+b}$

② if $a+b=1$ → (a) $2^1 = 2$ $F_1 = 2F_0$

if $a+b=2$ → (b) $2^2 = 4$ $F_1 = 4F_0$

$\underline{\underline{a+b}} = 2/3 < 1$

$F(K,L) = 10K^{1/3}L^{1/3}$

if $t=2.$ $F(tK,tL) = F(2K,2L) = 10(2K)^{1/3}(2L)^{1/3} = 2^{1/3+1/3}F(K,L)$

$= 2^{2/3} = 1.59 < 2$

$= F(200, 50) = 10(200)^{1/3}(50)^{1/3} = 1.59 * 500$

$= 795$

$F(K,L) = \underline{10K^{1/2}L^{1/2}} \Rightarrow F(100,25) = 500.$

$F(2K,L) = 10(\underbrace{200}_{2*K})^{1/2}(25)^{1/2} = 707$

if $t=2$

$F(tK,L) = 10(tK)^{1/2}L^{1/2} = t^{1/2}F(K,L) = \underline{\underline{707}}$

if $t=2$

$\boxed{\sqrt{2}}$

---

e.g. $f(x,y) = xy^2$, find $f(a+h,b)$, $f(a,b+k) - f(a,b)$

$$f(a+h, b) = (a+h)(b^2) = ab^2 + b^2h$$

$$f(a, b+k) - f(a,b) = a(b+k)^2 - ab^2 = a(b^2+2bk+k^2) - ab^2$$

$$= 2abk + ak^2$$

Partial Derivatives with Two Variables

$$Z = f(x,y)$$

Partial derivative is taken by keeping one variable constant.

$$\left.\frac{\partial Z}{\partial X}\right|_{\boxed{y=\bar{y}}} = \frac{\partial f(x,y)}{\partial x}$$

Notation

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = z'_x = f'_x(x,y)$$

$$= f'_1(x,y)$$

$$z = f(x,y)$$

e.g. $z = x^3 + 2y^2$ Find partial derivatives.

$$\left.\frac{\partial z}{\partial x}\right|_{y=\bar{y}} = 3x^2$$

$$\left.\frac{\partial z}{\partial y}\right|_{x=\bar{x}} = 4y$$

e.g. Find the partial derivatives of $f(x,y) = \frac{xy}{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2} = \frac{x^2y + y^3 - 2x^2y}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x(x^2+y^2) - xy(2y)}{(x^2+y^2)^2} = \frac{x^3 + xy^2 - 2xy^2}{(x^2+y^2)^2} = \frac{x^3 - xy^2}{(x^2+y^2)^2}$$

e.g. Given $X = A\frac{m^{\boxed{2.08}}}{p^{\boxed{1.5}}}$, (a) find partial derivatives, (b) Derive elasticities and interpret the results. X: milk consumption, m: income, p: price of milk, A: positive constant.

$$\frac{\partial X}{\partial p} = -1.5 A m^{2.08} p^{-2.5} \Leftarrow \text{if } p \uparrow \text{ by } \underline{1 \text{ unit}},\ X \downarrow \text{ by } 1.5 A m^{2.08} p^{-2.5}\ \underline{\text{unit}}$$

$$\frac{\partial X}{\partial m} = 2.08 A m^{1.08} p^{-1.5} \Leftarrow \text{if } m \uparrow \text{ by } \underline{1 \text{ unit}},\ X \uparrow \text{ by } 2.08 A m^{1.08} p^{-1.5}\ \underline{\text{unit}}$$

price elast. demand

$$\varepsilon_d^p = \frac{\partial X}{\partial p} \cdot \frac{p}{X} = -1.5 A m^{2.08} p^{-2.5}\, p \Big/ A m^{2.08} p^{-1.5} = \boxed{-1.5}$$

meaning?

$\Rightarrow$ 1% $\uparrow$ in p $\rightarrow$ 1.5% $\downarrow$ in X. Very important!

Income elas. demand

$$\varepsilon_d^m = \frac{\partial X}{\partial m} \cdot \frac{m}{X} = 2.08 A m^{1.08} p^{-1.5} \cdot \frac{m}{A m^{2.08} p^{-1.5}} = 2.08 \Rightarrow$$

1% $\uparrow$ in m $\rightarrow$ 2.08% $\uparrow$ in X

%

e.g. $F(K, L) = 10K^{1/2}L^{1/2}$

L = 100
K = 50

$$\frac{\partial F}{\partial K} = \frac{10}{2}K^{-1/2}L^{1/2} = 5\left(\frac{L}{K}\right)^{1/2} :$$

If K $\uparrow$ by 1 unit $\rightarrow$ $\uparrow$ F by $5\left(\frac{L}{K}\right)^{1/2}$

marginal product of capital

$$\frac{\partial F}{\partial L} = \frac{10}{2}K^{1/2}L^{-1/2} = 5\left(\frac{K}{L}\right)^{1/2} :$$

marginal product of labor

Higher Order Partial Derivatives $f(x,y)$

1st der. $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

2nd derivative $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$ $\quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

$$\boxed{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}}$$ ✓

e.g. $f(x, y) = 5x^4y^2 - 2xy^5$, find 1st and 2nd derivatives.

$$\frac{\partial f}{\partial x} = 20x^3y^2 - 2y^5 \qquad \frac{\partial f}{\partial y} = 10x^4y - 10xy^4$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 60x^2y^2 \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 10x^4 - 40xy^3$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 40x^3y - 10y^4 \overset{\checkmark}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 40x^3y - 10y^4$$

e.g. $F(K, L) = 10K^{1/2}L^{1/2}$, find 2nd derivative. Consider the meaning of the derived expression.

⊕ → $\frac{\partial F}{\partial K} = 5K^{-1/2}L^{1/2}$ $\qquad$ $\frac{\partial F}{\partial L} = 5K^{1/2}L^{-1/2}$

K↑ → F↑

⊖ → $\boxed{\frac{\partial^2 F}{\partial K^2}} = -\frac{5}{2}K^{-3/2}L^{1/2}$ $\qquad$ $\frac{\partial^2 F}{\partial L^2} = -\frac{5}{2}K^{1/2}L^{-3/2}$

Diminishing marginal productivity

e.g. $f(x, y) = x^3e^{y^2}$. Find 1st and 2nd derivatives and evaluate at (x,y)=(1,1)

$\frac{\partial f}{\partial x} = \boxed{3x^2e^{y^2}} = 3e$ $\qquad$ $\frac{\partial f}{\partial y} = 2yx^3e^{y^2} = 2e$

$\frac{\partial^2 f}{\partial x^2} = 6xe^{y^2} = 6e$ $\qquad$ $\frac{\partial^2 f}{\partial y^2} = 2x^3e^{y^2} + 4y^2x^3e^{y^2} = 2e + 4e = 6e$

→ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 6yx^2e^{y^2} = 6e$

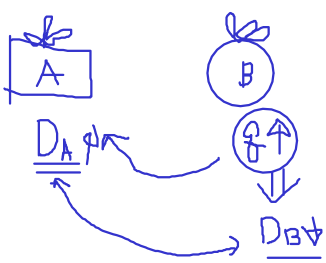
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ //

e.g. $D_A(p, q) = a - bp_A q_B^{-\alpha}$ where D: the quantity demanded for good A, p: the price of a product by firm A (own good), q: the price of a product by firm B (other related good) and a>0, b>0, 0<α<1. Find 1st derivatives with respect to p and q.

$\frac{\partial D}{\partial p} = -bq^{-\alpha}$ $\qquad$ p↑ by 1 unit, D↓ by $bq^{-\alpha}$ unit.

$\frac{\partial D}{\partial q} = \alpha bpq^{-\alpha-1}$ $\qquad$ q↑ by 1 unit, D↑ by $\alpha bpq^{-\alpha-1}$ unit

⇒ Good A & B are (substitute.)



Functions of More Variables

$f(\mathbf{x}) = f(x_1 \cdots \cdots x_n)$

vector

independent

e.g. $\boxed{\text{Housing price}}$ Y $= f($ size m^2, #rooms, #bathroom, location, access transportation, parking, prox to park, schools, beach/sea, view, direction .. floor, elevator, ... $)$

$Y = \beta_0 + \beta_1 \underset{\#room}{X_1} + \beta_2 X_2 + \cdots + \beta_n X_n$

$\frac{\partial Y}{\partial X_1}$ ⇐ # room ↑ by $\frac{1 \text{ unit}}{1 \text{ unit}}$ ⇒ price ↑ by $\frac{\partial Y}{\partial X_1} \frac{TL}{\text{unit}}$.

e.g. Linear Function

$\beta_1 X_1 + \beta_2 X_2 + \cdots$

Cobb-Douglas Function

$F(X_1 \cdots X_n) = A X_1^{a_1} \cdot X_2^{a_2} X_3^{a_3} \cdots X_n^{a_n}$

$\Rightarrow \ln F(X_1 \cdots X_n) = \ln A + a_1 \ln X_1 + a_2 \ln X_2 \cdots + a_n \ln X_n$

e.g. $y = 3x_1^{0.015} x_2^{0.25} x_3^{0.35}$

output of milk (y), L (x_1), grass consumption (x_2), K (x_3)

→ If all the factors of productions are doubled, what will happen to y?

$y(2X_1, 2X_2, 2X_3) = 3(2X_1)^{0.015}(2X_2)^{0.25}(2X_3)^{0.35}$

$= 2^{0.015+0.25+0.35} \cdot y$

$a+b+c<1$

$= 2^{0.615} y \longrightarrow 1.53\, y$

Partial Derivatives with More Variables

$$z = f(\mathbf{x}) = f(x_1, \ldots x_n)$$

$$\frac{\partial z}{\partial x_i} \Big|_{\text{Keep everything else constant}} = \frac{\partial f}{\partial x_i}$$

e.g. $f(x_1, x_2, x_3) = 10x_1^3 + 2x_2^5 + 3\cancel{3}_3^2$ (x). Find 1st partial derivatives.

$$\frac{\partial f}{\partial x_1} = 30x_1^2$$

$$\frac{\partial f}{\partial x_2} = 10x_2^4$$

$$\frac{\partial f}{\partial x_3} = 6x_3$$

Higher-order Partial Derivatives

- Second-order partials

$$f(x_1, x_2, x_3) \Rightarrow 3 \times 3 \quad \underline{\underline{\text{Hessian Matrix}}}$$

$$f''(\mathbf{x}) = \begin{pmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{pmatrix}$$

$$\boxed{f''_{ij} = f''_{ji} \quad \text{for } i \neq j}$$

$$f(x_1 \ldots x_n) \Rightarrow n \times n \quad \text{Hessian Matrix}$$

e.g. $f(x_1, x_2, x_3) = 5x_1^2 + x_1x_2^3 - x_2^2x_3^2 + x_3^3$. Find Hessian Metrix.

$$\begin{cases} f_1' = 10X_1 + X_2^3 \\ f_2' = 3X_1X_2^2 - 2X_2X_3^2 \\ f_3' = -2X_2^2X_3 + 3X_3^2 \end{cases}$$

$$H = \begin{pmatrix} f_{11}'' = 10 & f_{12}'' = \underline{3X_2^2} & 0 \\ \underline{3X_2^2} & f_{22}'' = 6X_1X_2 - 2X_3^2 & f_{23}'' = \underline{-4X_2X_3} \\ \underline{\underline{0}} & -4X_2X_3 & f_{33}'' = -2X_2^2 + 6X_3 \end{pmatrix}$$

$f_{ij}'' = f_{ji}''$ ⇐ young's theorem

Young's Theorem

$$\frac{\partial}{\partial X_i}\left(\frac{\partial f}{\partial X_j}\right) = \frac{\partial}{\partial X_j}\left(\frac{\partial f}{\partial X_i}\right)$$

e.g. $f(x_1, x_2, x_3) = 3x_1x_2x_3 + x_1^2x_2 - x_1x_3^3$. Find Hessian Metrix and evaluate at f(1,0,1).

Partial Elasticities

e.g.

- (Own) price elasticity of demand

- Income Elasticity of Demand

- Cross price elasticity of demand

e.g. Find elasticity of z with respect to x when (1) $z = Ax^a y^b$, (2) $z = xye^{x+y}$.

e.g. $D = Ap^{-0.28}m^{0.34}$. Find (a) price elasticity of demand and (b) income elasticity of demand. Interpret the result.

e.g. $D_i = Am^{\alpha}{p_i}^{-\beta}p_j^{\gamma}$. Find (a) own price elasticity, (b) cross price elasticity and (c) income elasticity of demand. Interpret the results.