ECO106/138 Week 5 Lecture Note Template

Systems of Linear Equations

$$\begin{cases} x_1 + 2x_2 = \boxed{2} \\ 2x_1 - x_2 = \boxed{4} \end{cases}$$

Solve for x_1^* and x_2^* by using matrix.

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}}_{\substack{A \\ 2\times 2}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\substack{x \\ 2\times 1}} = \underbrace{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}_{\substack{b \\ 2\times 1}}$$

$$\underset{2\times 2 \; 2\times 1}{AX} = \begin{pmatrix} \boxed{x_1 + 2x_2} \\ \boxed{2x_1 - x_2} \end{pmatrix}_{2\times 1}$$

$AB \neq BA$

$\cancel{\underset{2\times 1}{b}\;\underset{2\times 2}{A}}$

$$\boxed{\begin{aligned} Ax &= b \\ x &= A^{-1}b \end{aligned}}$$

$$A^{-1}AX = A^{-1}b$$

pre multiple

$$\boxed{\underset{2\times 1}{X} = \underset{2\times 2}{A^{-1}}\,\underset{2\times 1}{b}}$$

(0) Write eq. in matrix format

(1) Calculate A^{-1}

(2) Done $A^{-1}b$

~~Generally,~~

eg. $\underset{A}{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}} \underset{X}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \underset{b}{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}$ $\quad \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} \; \end{pmatrix}$

$$A^{-1} = \frac{1}{-5}\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} \checkmark \qquad X = A^{-1}b = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$|D| = -1 - 4 = -5$$

$$\underset{2\times 1}{X} = \begin{pmatrix} \frac{1}{5} * 2 + \frac{2}{5} * 4 \\ \frac{2}{5} \times 2 - \frac{1}{5} * 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} + \frac{8}{5} \\ \frac{4}{5} - \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$$

Generally,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\Downarrow$$

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{pmatrix}} \underset{X}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \underset{b}{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}} \Rightarrow \boxed{X = A^{-1}b}$$

e.g.

$$2x_1 - 2x_2 + x_3 = 3$$

$$3x_1 + x_2 - x_3 = 7$$

$$x_1 - 3x_2 + 2x_3 = 0$$

Solve for x_1^* , x_2^* and x_3^* by using matrix

①
$$\underbrace{\begin{pmatrix}2 & -2 & 1\\ 3 & 1 & -1\\ 1 & -3 & 2\end{pmatrix}}_{\substack{A\\3\times 3}}\underbrace{\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}}_{\substack{X\\3\times 1}} = \underbrace{\begin{pmatrix}3\\ 7\\ 0\end{pmatrix}}_{\substack{b\\3\times 1}} \Rightarrow \overset{3\times 1}{\boxed{X}} = \overset{3\times 3}{A^{-1}}\overset{3\times 1}{b}$$

①

$$|A| = 2(2-3)+2(6+1)+(-9-1) = -2+14-10 = 2$$

$$\text{adj}A = \begin{pmatrix}(2-3) & -(6+1) & (-9-1)\\ -(-4+3) & (4-1) & -(-6+2)\\ (2-1) & -(-2-3) & (2+6)\end{pmatrix}' = \begin{pmatrix}-1 & -7 & -10\\ 1 & 3 & 4\\ 1 & 5 & 8\end{pmatrix}' = \begin{pmatrix}-1 & 1 & 1\\ -7 & 3 & 5\\ -10 & 4 & 8\end{pmatrix}$$

$$A^{-1} = \begin{pmatrix}-1/2 & 1/2 & 1/2\\ -7/2 & 3/2 & 5/2\\ -10/2 & 4/2 & 8/2\end{pmatrix}$$

②
$$\underbrace{X}_{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}} = A^{-1}b = \begin{pmatrix}-1/2 & 1/2 & 1/2\\ -7/2 & 3/2 & 5/2\\ -5 & 2 & 4\end{pmatrix}\begin{pmatrix}3\\ 7\\ 0\end{pmatrix}$$

$$X = \begin{pmatrix}-\frac{1}{2}\cdot 3+\frac{1}{2}\cdot 7\\ -\frac{7}{2}\cdot 3+\frac{3}{2}\cdot 7\\ \underset{-15}{-5\cdot 3} + \underset{14}{2\cdot 7}\end{pmatrix} = \begin{pmatrix}2\\ 0\\ -1\end{pmatrix} = \begin{pmatrix}x_1^*\\ x_2^*\\ x_3^*\end{pmatrix}$$

Cramer's Rule ("easier" way to solve x*. No need to derive an inverse matrix.)

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$$
$$a_{21}x_1 + \cdots\cdots a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + \cdots\cdots a_{nn}x_n = b_n$$

$$\underset{\substack{A\\n\times n}}{\begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & & \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{pmatrix}} \underset{\substack{X\\n\times 1}}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \underset{\substack{b\\n\times 1}}{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}$$

$$\begin{array}{c} \\ |D_j| = \end{array} \begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad j^{th} \\ \begin{vmatrix} a_{11} & a_{12} \cdots a_{1,j-1} & b_1 & a_{1n} \\ \vdots & \ddots & b_2 & \\ & & \vdots & \\ a_{n1} & & b_n & a_{nn} \end{vmatrix} \end{array}$$

$$|D_1| = \begin{vmatrix} b_1 & a_{12} \cdots & a_{1n} \\ b_2 & \ddots & \\ \vdots & & \\ b_n & & a_{nn} \end{vmatrix}$$

$$x_1 = \frac{|D_1|}{|A|}, \quad x_2 = \frac{|D_2|}{|A|}, \quad x_3 = \frac{|D_3|}{|A|}$$

e.g.

$$7x_1 - x_2 - x_3 = 0$$
$$10x_1 - 2x_2 + x_3 = 8$$
$$6x_1 + 3x_2 - 2x_3 = 7$$

Q1 Solve for x_1^*, x_2^* *and* x_3^* by using Cramer's Rule

$$b = \begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

HW Q2. Use inverse matrix to solve for x_1^*, x_2^* *and* x_3^*.

$$A = \begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{pmatrix}$$

$$|A| = 7(4-3) + 1(-20-6) - 1(30+12)$$
$$= 7 - 26 - 42 = -61$$

$$x_1 = \frac{|D_1|}{|A|} = \frac{-61}{-61} = 1$$

$$D_1 = \begin{pmatrix} 0 & -1 & -1 \\ 8 & -2 & 1 \\ 7 & 3 & -2 \end{pmatrix}$$

$$|D_1| = 1(-16-7) - 1(24+14) = -23 - 38 = -61$$

$$D_2 = \begin{pmatrix} 7 & 0 & -1 \\ 10 & 8 & 1 \\ 6 & 7 & -2 \end{pmatrix}$$

$$|D_2| = 7(-16-7) - 1(70-48) = -183 \qquad x_2 = \frac{-183}{-61} = 3$$

$$D_3 = \begin{pmatrix} 7 & -1 & 0 \\ 10 & -2 & 8 \\ 6 & 3 & 7 \end{pmatrix}$$

$$|D_3| = \overbrace{7(-14-24)}^{-266} + \overset{-1(-1)}{1}\overbrace{(70-48)}^{22} = -244 \qquad x_3 = \frac{-244}{-61} = 4$$

$$x^* = \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

e.g.

$$8x_1 - x_2 = 16$$
$$2x_2 + 5x_3 = 5$$
$$2x_1 + 3x_3 = 7$$

$$8X_1 - X_2 + \boxed{} = 16$$
$$\boxed{} + 2X_2 + 5X_3 = 5$$
$$2X_1 + \boxed{} + 3X_3 = 7$$

Q1 Solve for x_1^* , x_2^* and x_3^* by using Cramer's Rule

HW Q2. Use inverse matrix to solve for x_1^* , x_2^* and x_3^* .

$$A = \begin{pmatrix} 8 & -1 & 0 \\ 0 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 16 \\ 5 \\ 7 \end{pmatrix}$$

$$|A| = 8(6) + 1(-10) = 48 - 10 = 38$$

$$|D_1| = \begin{vmatrix} 16 & -1 & 0 \\ 5 & 2 & 5 \\ 7 & 0 & 3 \end{vmatrix} = 16(6) + (15 - 35) = 76 \qquad X_1 = \frac{76}{38} = 2$$

$$|D_2| = \begin{vmatrix} 8 & 16 & 0 \\ 0 & 5 & 5 \\ 2 & 7 & 3 \end{vmatrix} = 8(15 - 35) - 16(-10) = 0 \qquad X_2 = 0 \qquad X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$|D_3| = \begin{vmatrix} 8 & -1 & 16 \\ 0 & 2 & 5 \\ 2 & 0 & 7 \end{vmatrix} = 8(14) + 2(-5 - 32) = 38 \qquad X_3 = 1$$

Leontief Model

Output of any industry is needed as an input in any other industries, or even for that industry itself.

⇨ What is the “correct” level of output to meet the input requirements of other industries?

e.g.

Electricity => used within power plant + Car company + Steel company + Coal company+….+HOUSEHOLD

Steel => Power plant +Car company+ Steel company + coal company +…+ HOUSEHOLD

Auto => Power plant + Car company + Steel company + Coal company + …. + HOUSEHOLD

….

< Input Output Table >

Input-Output Table

$a_{i,j}$: Input coefficient (i = input, j = output) => $\$a_{i,j}$ worth of input I is needed to produce \$1 worth of output j.

$a_{23} = a_{steel, elec.}$

$$\begin{array}{c|ccc} & & \text{Output} & \\ \text{Input} & 1.\text{Auto} & 2.\text{Steel} & 3.\text{Elec.} \\ \hline 1.\text{Auto} & a_{11} & a_{12} & a_{13} \\ 2.\text{Steel} & a_{21} & a_{22} & a_{23} \\ 3.\text{Elec.} & a_{31} & a_{32} & a_{33} \end{array}$$

input output

$\$a_{23}$ worth of steel is used to produce \$1 worth of Elec.

0.1

$a_{13}+a_{23}+a_{33}$ total cost of producing \$1 worth of Elec. (1)

$\$a_{31}$ worth of input 3 (electricity) needed to produce \$1 of Auto

0.2

e.g. Steel industry uses

0.1 a_{12}
+
0.2 a_{22}
+
0.3 a_{32}

0.6 to produce \$1 worth of steel
<\$1

$\sum_{i=1}^{3} a_{i2} < 1$ output

e.g. Assume that there are 3 sectors; 1. Auto, 2. Steel and 3. Electricity

Input-output matrix is defined as:

$$A = \begin{array}{c} \\ \text{Auto} \\ \text{Steel} \\ \text{Elect} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Auto} & \text{Steel} & \text{Elect.} \end{array} \\ \begin{bmatrix} 0.15 & 0.1 & 0.05 \\ 0.40 & 0.2 & 0.1 \\ 0.10 & 0.25 & 0.2 \end{bmatrix} \end{array}$$

Steel column: 0.1 + 0.2 + 0.25 = \$0.55 used as inputs to produce \$1 of (S).

Elect row: 0.10 + 0.25 + 0.2 = 55¢ worth of elect. used as input in 3 sectors.

10¢ worth of steel used to produce \$1 of (E).

10¢ worth of (E) used to produce \$1 of Auto

Q1.Steel industry uses \$(0.55) worth of input to produce \$1 worth of steel.

Q2. \$(0.55) worth of Electricity is used as input in 3 sectors.

Now let's define: X_1: total value of Auto

X_j: total value (\$) of good j that industry j is going to produce in a certain year.

$a_{i,j}$: \$ worth of good i required for producing \$1 worth of good j.

$a_{i,j}X_j$: total value (\$) of good I need to produce $\$X_j$ worth of good j.

d_j: *Final Demand*: Good j directly consumed by households.

e.g. For input 1 (Auto) X_3 + d (1. Auto, 2. Steel, 3. Electricity)

$$A = \begin{array}{cc} & \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \\ a_1 & \begin{bmatrix} 0.15 & 0.10 & 0.05 \\ 0.40 & 0.20 & 0.10 \\ 0.10 & 0.25 & 0.20 \end{bmatrix} \end{array} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

\$ 0.15 → \$ 1
\$ 15 → \$ 100
E \$ 0.10 → \$ 1 A
E \$ 10 ← \$ 100 A, X_3 1000

What is the meaning of "$a_{1,1}X_1 + a_{1,2}X_2 + a_{1,3}X_3$ "?

total value of Auto used in 3 sect.

Find X_1^*, X_2^*, X_3^* to meet all the demand? own, other sectors, HH

Total Production of 1st Good (Auto) Demand for Auto

$$\Rightarrow X_1 = \underbrace{a_{11}X_1}_{\text{Auto}} + \underbrace{a_{12}X_2}_{\text{steel}} + \underbrace{a_{13}X_3}_{\text{Elec}} + d_1$$

Final demand.

Total Production of 2nd good (Steel)

$$\Rightarrow \underbrace{X_2}_{\text{output}} = \underbrace{a_{21}X_1}_{\text{auto industry uses}} + a_{22}X_2 + a_{23}X_3 + d_2$$

\$$a_{21}X_1$ worth of steel as their input.

Total Production of 3rd good (Electricity)

$$\Rightarrow X_3 = a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + d_3$$

$$X = \underbrace{\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}} \overset{X}{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}} + \overset{d}{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$X = AX + d$ ⇒ want to solve for X

$IX - AX = d$

$(I-A)\boxed{X} = d$

$$\boxed{X = (I-A)^{-1}d}$$

Before

$Ax = b$

$x = A^{-1}b$

⇨ Express everything discussed so far in Matrix Format!

$$X = AX + d$$
$$IX - AX = d$$
$$(I - A)X = d$$

$$\boxed{X = (I - A)^{-1} d}$$

input output table (A); final demand (d)

e.g. Given $A = \begin{bmatrix} 0.15 & 0.10 & 0.05 \\ 0.40 & 0.20 & 0.10 \\ 0.10 & 0.25 & 0.20 \end{bmatrix}$, $d = \begin{bmatrix} 10 \\ 5 \\ 20 \end{bmatrix}$, Find X* (1) using Cramer's Rule, (2) using inverse matrix.

input-output table; final demand.

① $(I-A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.15 & 0.1 & 0.05 \\ 0.4 & 0.2 & 0.1 \\ 0.1 & 0.25 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.85 & -0.1 & -0.05 \\ -0.4 & 0.8 & -0.1 \\ -0.1 & -0.25 & 0.8 \end{pmatrix}$

HW

Inverse ② $(I-A)^{-1}d = \begin{bmatrix} 1.279 & 0.1924 & 0.104 \\ 0.6864 & 1.404 & 0.2184 \\ 0.3744 & 0.4628 & 1.3313 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 20 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

Cramer ② $|D| = |I - A| = 0.48075$

$|D_1| = |(I-A)_1| = \begin{vmatrix} 10 & -0.1 & -0.05 \\ 5 & 0.8 & -0.1 \\ 20 & -0.25 & 0.8 \end{vmatrix} = 7.6125$

$|D_2| = |(I-A)_2| = \begin{vmatrix} 0.85 & 10 & -0.05 \\ -0.4 & 5 & -0.1 \\ -0.1 & 20 & 0.8 \end{vmatrix} = 8.775$

$|D_3| = |(I-A)_3| = \begin{vmatrix} 0.85 & -0.1 & 10 \\ -0.4 & 0.8 & 5 \\ -0.1 & -0.25 & 20 \end{vmatrix} = 15.7125$

$X_1 = \dfrac{|(I-A)_1|}{|I-A|}$

$X_2 = \dfrac{|(I-A)_2|}{|I-A|}$

$X_3 = \dfrac{|(I-A)_3|}{|I-A|}$

$X_1 = \dfrac{7.6125}{0.48075} = 15.83$

$X_2 = \dfrac{8.775}{0.48075} = 18.25$

$X_3 = \dfrac{15.7125}{0.48075} = 32.682$