ECO106/138 Week 4 Lecture Note Template

Basic Rules for Determinants

1. The interchange of lows and columns does not affect the value of a determinant.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow A' = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

$$|A| = a_{11}a_{22} - a_{12}a_{21} = |A'| = a_{11}a_{22} - a_{12}a_{21}$$

$$|A| = |A'|$$

2. The interchange of any two rows (or any two columns) will alter the sign, but not the numerical value of the determinant.

$$A_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow A_1 = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$$

$$|A_0| = a_{11}a_{22} - a_{12}a_{21}$$

$$|A_1| = a_{12}a_{21} - a_{11}a_{12}$$
$$= -(|A_0|)$$

$$A_2 = \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix}$$

$$|A_2| = a_{12}a_{21} - a_{11}a_{22}$$
$$= -(|A_0|)$$

e.g. $|A_0| = \begin{vmatrix} \boxed{0 \quad 1 \quad 3} \\ 2 \quad 5 \quad 7 \\ 3 \quad 0 \quad 1 \end{vmatrix}$ $\qquad |A_1| = \begin{vmatrix} 3 \quad 1 \quad 0 \\ 7 \quad 5 \quad 2 \\ \boxed{1 \quad 0 \quad 3} \end{vmatrix}$

even $\qquad$ 1d odd

$$|A_0| = -1(2-21)+3(-15)$$
$$= 19 - 45$$
$$= -26$$

$$|A_1| = 1(2)+3(15-7)$$
$$= 2 + 3 \cdot 8$$
$$= 26$$

3. The multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold.

$$A_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad A_1 = \begin{pmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$|A_0| = a_{11}a_{22} - a_{12}a_{21} \qquad |A_1| = k(a_{11}a_{22}) - k(a_{12}a_{21}) = k(a_{11}a_{22} - a_{12}a_{21}) = k\,|A_0|$$

e.g.

$$A_0 = \begin{pmatrix} 2 & 0 \\ 3 & 5 \end{pmatrix} * 2 \qquad A_1 = \begin{pmatrix} 2 & 0 \\ 6 & 10 \end{pmatrix} \quad \Rightarrow \quad |A_1| = \underline{\underline{2 * 10 = 20}}$$

$$|A_0| = 10 \qquad = 20$$

k $\boxed{1st}$ → 2nd row

4. The addition (subtraction) of a multiple of any row (column) to (from) another row (column) will leave the value of the determinant unaltered.

$$A_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \leftarrow \text{add } k * \text{1st row} \qquad |A_0| = a_{11}a_{22} - a_{12}a_{21}$$

$$\|$$

$$A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} + ka_{11} & a_{22} + ka_{12} \end{pmatrix} \Rightarrow |A_1| = a_{11}a_{22} - a_{12}a_{21}$$

e.g. $|A_0| = \begin{vmatrix} \boxed{2 \quad 3} \\ 3 \quad 5 \end{vmatrix}$ $*k$ $2 + (*2)$ $= 10 - 9 = 1$

$$|A_1| = \begin{vmatrix} 2 & 3 \\ 3 + 2(2) & 5 + 2*3 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 7 & 11 \end{vmatrix} = 22 - 21 = 1$$

a_{11} a_{12}

k k

$k = 5$

$$A_0 = \begin{pmatrix} 5 & 2 \\ 6 & 10 \end{pmatrix} \quad *5$$

$$A_1 = \begin{pmatrix} 5 + 5 \cdot 6 & 2 + 5 \cdot 10 \\ 6 & 10 \end{pmatrix} = \begin{pmatrix} 35 & 52 \\ 6 & 10 \end{pmatrix}$$

$|A_0| = 38 \quad = \quad |A_1| = 38$

5. If one row (or column) is a multiple of another row (column), the value of the determinant will be zero.

$$A = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix} \qquad \text{e.g. } A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} \ 3 * \text{1st row}$$

$$|A| = kab - kab = 0 \qquad |A| = 18 - 18 = 0$$

2×2 $\quad 4\times 4$

3×3 $\quad \vdots$

6. The determinant of the product of two nxn matrices A and B is the product of the determinants of each of the factors.

RHS

$$|AB| = |A|\cdot|B|$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A\cdot B = \begin{bmatrix} \boxed{ae+bg} & \boxed{af+bh} \\ \boxed{ce+dg} & \boxed{cf+dh} \end{bmatrix}$$

$$B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$|AB| = (ae+bg)(cf+dh) - (af+bh)(ce+dg)$$

$$= a\cancel{cef} + adeh + bcfg + \cancel{bdgh} - a\cancel{cef} - adfg - bceh - \cancel{bdgh}$$

$$= adeh + bcfg - adfg - bceh$$

$$|A||B|$$

$$[ad - bc][eh - fg]$$

7. If α is a real number, $|\alpha A| = \alpha^n|A|$ where n is the number of row and column. e.g. 2 x 2 matrix => n= 2

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad |A| = a_{11}a_{22} - a_{12}a_{21}$$

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix}$$

$$|\alpha A| = \alpha^{\boxed{2}} a_{11}a_{22} - \alpha^2 a_{12}a_{21} = \alpha^2 |A|$$

e.g. $|A_0| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-2) - 2(-2) + 3(-1)$

$= 4 - 3 = +1$

$\alpha = 2$ $\quad |A_1| = \begin{vmatrix} 2 & 4 & 6 \\ 0 & 2 & 4 \\ 2 & 2 & 4 \end{vmatrix} \Rightarrow \boxed{2^3 |A_0| = 8}$ rule suggests

confirm

$= 2(8-8) + 2(16-12) = 8$ ✓

~~If A~~ is not a square matrix, does this work?

$|A_0| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix}$ $\quad \alpha = 2$ $\quad |A_1| = \begin{vmatrix} 2 & 4 & 6 \\ 0 & 2 & 4 \end{vmatrix}$ ←

2×3

We can't calculate determinant of non-square matrix $(m\times n)$ $m \neq n$!

Inverse of a Matrix

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$a \rightarrow \frac{1}{a}$

e.g. $\begin{pmatrix} a & b \\ 2a & 2b \end{pmatrix} \Rightarrow$ No inverse.

$|A| \neq 0 \Leftrightarrow A$ has an inverse.

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$(-1)^{i+j}$

cofactor

$$\underbrace{\text{adj } A}_{\text{Adjoint}} = \underset{\substack{n \times n \\ \text{cofactor matrix}}}{C^{T}} = \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{21}| & |C_{22}| & \cdots & |C_{n2}| \\ \vdots & & \ddots & \vdots \\ \vdots & & & |C_{nn}| \end{bmatrix}$$

e.g.

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \quad \text{Find } A^{-1}.$$

① Find $|A| = 1(1-4) + 3(8-5) = -3 + 9 = 6$

② adj A

$i+j$ odd

$$\text{adj } A = \begin{bmatrix} \overset{|C_{11}|}{6} & \overset{|C_{12}|}{-(15-1)} & \overset{|C_{13}|}{-2} \\ -3 & 10 & -(-1) \\ -3 & \underset{|C_{32}|}{-6} & 3 \end{bmatrix}$$

cofactor

$$|C_{ij}| = (-1)^{(i+j)} |M_{ij}|$$

if $i+j$ = odd $\Rightarrow (-1)$

$|M_{ij}| =$

$$|M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 6 & -3 & -3 \\ -14 & 10 & 6 \\ -2 & 1 & 3 \end{bmatrix}$$

$6/6 \quad 3/6$

3. $$A^{-1} = \frac{1}{6} \begin{bmatrix} 6 & -3 & -3 \\ -14 & 10 & 6 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -2.33 & 1.67 & 1 \\ -0.33 & 0.167 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} \overset{①}{a_{11}} & \overset{②}{a_{12}} \\ \overset{③}{a_{21}} & \overset{④}{a_{22}} \end{bmatrix}$$

e.g. $\boxed{2 \times 2}$

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \qquad \text{Find } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$|A| = -2$

$$\text{adj } A = \begin{bmatrix} 0 & \overset{a_{12}}{-1} \\ \underset{a_{21}}{-2} & 3 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{-2} \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1/2 & -3/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.5 & -1.5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 5 \\ 10 & 3 \end{bmatrix} \Rightarrow \text{Find } A^{-1}$$

1. $|A| = 6 - 50 = -44$.
2. $\text{adj} A = \begin{bmatrix} 3 & -10 \\ -5 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -5 \\ -10 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{-44}\begin{bmatrix} 3 & -5 \\ -10 & 2 \end{bmatrix} = \begin{bmatrix} 3/-44 & 5/44 \\ 10/44 & 2/44 \end{bmatrix}$$

eg.

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix} \quad \text{Find } A^{-1}$$

Report

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

~ chat.

$$|A| = \sum a_{ij} |C_{ij}|$$

$$|A| = 4\begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 4(21) + 3(2+3) = 84 + 15 = 99$$

$$A^{-1} = \frac{1}{99}\begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

challenge

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 2 & 1 & -1 & 3 \\ 1 & 3 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

A^{-1}

$|C_{11}|$

$$\text{adj} A = \begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix}'$$

$$= \begin{bmatrix} 21 & +6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}' = \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

$$A^{-1} = \frac{1}{99}\begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 0.212 & -0.07 & 0.05 \\ 0.06 & 0.313 & -0.08 \\ -0.09 & 0.03 & 0.121 \end{bmatrix}$$

$$= \begin{bmatrix} 21/99 & -7/99 & 5/99 \\ 6/99 & 31/99 & -8/99 \\ -9/99 & 3/99 & 12/99 \end{bmatrix}$$

ok for exam.

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 2 & 1 & -1 & 3 \\ 1 & 3 & 2 & -1 \\ \boxed{0 \quad 0 \quad 1 \quad 0} \end{bmatrix}$$

a_{43}

$$|A| = -1 \begin{vmatrix} \boxed{5 \quad 0 \quad 0} \\ 2 \quad 1 \quad 3 \\ 1 \quad 3 \quad -1 \end{vmatrix} = -1\left(5 \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}\right)$$

$$= -1 \cdot 5(-1-9)$$

$$= 50 \,//$$

$$\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix} 1 & -1 & 3 \\ 3 & 2 & -1 \\ 0 & 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 0 & 1 & 0 \end{vmatrix} & \bigcirc & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc & \begin{vmatrix} 5 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix} \end{bmatrix}$$

4×4

C_{44}

$$= \begin{bmatrix} 10 & -5 & 0 & -5 \\ 0 & 5 & 0 & 15 \\ 0 & 15 & 0 & -5 \\ 0 & -25 & 50 & 25 \end{bmatrix}' = \begin{bmatrix} 10 & 0 & 0 & 0 \\ -5 & 5 & 15 & -25 \\ 0 & 0 & 0 & 50 \\ -5 & 15 & -5 & 25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{50} \operatorname{Adj} A$$

Properties of the Inverse

Let A and B be invertible n x n matrix, then,

(a) A^{-1} is invertible, $(A^{-1})^{-1} = A$

$(AB)' = B'A'$

(b) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$,

(c) A' is invertible, and $(A')^{-1} = (A^{-1})'$

(d) $(cA)^{-1} = c^{-1}A^{-1}, c \neq 0$

scaler $\frac{1}{c}$

$A = \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix}$ $B = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

Lets try (b) LHS RHS

$(AB)^{-1} = B^{-1} \cdot A^{-1}$

even odd

Systems of Linear Equations