Week 3 Lecture Note Template

CH.15 Matrix and Vector Algebra

CH.16 Determinants and Inverse Matrices

- Terminologies

column

entry

element

nth column

row 1

2

$$\underset{m \times n}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{pmatrix}$$

row column

main diagonal

# of rows

# of columns

mth row

nth column

Vector

1 row

$$\underset{1 \times n}{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \end{bmatrix}$$ row vector

$$\underset{m \times 1}{C} = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}$$ column vector

Matrix Operations

Addition & Rules

$$\underset{2 \times 2}{A} = \begin{pmatrix} 0 & 1 \\ \boxed{2} & 3 \end{pmatrix} \qquad \underset{2 \times 2}{B} = \begin{pmatrix} 1 & -1 \\ \boxed{5} & 2 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 0+1 & 1-1 \\ 2+5 & 3+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 7 & 5 \end{pmatrix}$$

$$5(A+B) = 5A + 5B$$

$$5\begin{pmatrix} 1 & 0 \\ 7 & 5 \end{pmatrix} = 5\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} + 5\begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 \\ 35 & 25 \end{pmatrix} \qquad \begin{pmatrix} 0 & 5 \\ 10 & 15 \end{pmatrix} + \begin{pmatrix} 5 & -5 \\ 25 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 \\ 35 & 25 \end{pmatrix}$$

Matrix Multiplications

$A_{m\times n} B_{n\times k}$ $A_{3\times 2} B_{2\times 3}$ ✓ $A_{2\times 3} B_{2\times 2}$ ✗

$$A_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B_{2\times 3} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$\underset{2\times 3}{AB} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}\cdot b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}\cdot b_{11} + a_{22}\cdot b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{pmatrix}$$

e.g. $A_{3\times 3} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & -1 & 6 \end{pmatrix}$ $B_{3\times 2} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} -1 & 2 \\ 8 & 5 \\ 5 & 14 \end{pmatrix}_{3\times 2}$$

$0\cdot 3 + 1\cdot 1 + 2\cdot(-1)$ $0\cdot 2 + 1\cdot 0 + 2\cdot 1$ $2\cdot 3 + 3\cdot 1 + 1\cdot(-1)$

$AB \neq BA$ — order is important

Rules for Matrix Multiplication

1. $(AB)C = A(BC) = ABC$
2. $A(B+C) = AB + AC$
3. $(A+B)C = AC + BC$
4. $(A+B)(A+B) = AA + \underline{AB + BA} + BB$ ($\neq AA + 2AB + BB$)
5. $A^2 = AA$
6. $AB \neq BA$

e.g. Using $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, show all the rules 1. ~ 6.

1. $(AB) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$

$(AB)C = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 9 \\ 7 & 5 \end{pmatrix}$

$6\cdot 1 + 3\cdot 2$ $6\cdot 1 + 3\cdot 1$ $3\cdot 1 + 2\cdot 2$ $3\cdot 1 + 2\cdot 1$

$\cancel{2\times 3 \; 3\times 4}$ 2×4

4. LHS: $(A+B)(A+B)$ = RHS: $AA + \underline{AB} + \underline{BA} + BB$

$A+B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$

$(A+B)(A+B) = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 12 & 12 \end{pmatrix}$ LHS

$AA = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$BA = \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & 8 \end{pmatrix}$

$BB = \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 6 & 1 \end{pmatrix}$

RHS $\begin{pmatrix} 4 & 4 \\ 12 & 12 \end{pmatrix} = AA + AB + BA + BB = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} + \underbrace{\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}}_{AB} + \underbrace{\begin{pmatrix} 0 & -1 \\ 3 & 8 \end{pmatrix}}_{BA} + \begin{pmatrix} -3 & -2 \\ 6 & 1 \end{pmatrix}$

- Identity Matrix

$$\underset{n\times n}{I_n} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & 1 & \ddots & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$\underset{2\times 2}{I_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{main diagonal} = 1$$

$$\underset{3\times 3}{I_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

off-diagonal = 0

$$\boxed{IA = AI = A}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad IA = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = A$$

$$AI = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = A$$

- Transpose ← switch row & columns

$$\underset{m\times n}{A} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \Big\} m \text{ rows} \;\Rightarrow\; \underset{n\times m}{A'} = \overbrace{\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{13} & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}}^{m \text{ columns}} \Big\} n \text{ rows}$$

n rows

nth row 1st col

n rows 2nd col

excel

f = transpose(:)

1990	1991	1992	...	...	...	2020
○	○	○				

→ paste as transpose

1990
1991
1992
⋮
2020

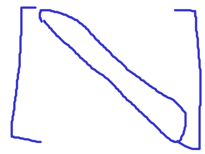
e.g. $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad A' = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$$\underset{2\times 4}{B} = \begin{pmatrix} 1 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

$$\underset{4\times 2}{B'} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$$

Rules:

1. $(A')' = A$ $\qquad \alpha$ = scaler = 2
2. $(A+B)' = A' + B'$
3. $(\alpha A)' = \alpha A'$
4. $(AB)' = B'A' \neq A'B'$
5. $A = A'$ if A is symmetric



e.g.

$$A = \begin{array}{cc} a_{11} & a_{12} \end{array} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{array}{cc} a_{21} & a_{22} \end{array} \qquad B = \begin{array}{cc} b_{11} & b_{12} \end{array} \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix} \begin{array}{cc} b_{21} & b_{22} \end{array}$$

LHS

$$(AB)'$$

$$A'B'$$

$$B'A'$$

$$AB = \begin{pmatrix} 1+8 & 3+10 \\ 3+16 & 9+20 \end{pmatrix} = \begin{pmatrix} 9 & 13 \\ 19 & 29 \end{pmatrix}$$

$$(AB)' = \begin{pmatrix} 9 & 19 \\ 13 & 29 \end{pmatrix} \quad \boxed{LHS}$$

$$\boxed{RHS}$$

$$A' = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad B'A' = \underbrace{\begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix}}_{B'} \underbrace{\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}}_{A'} = \begin{pmatrix} 1+8 & 3+16 \\ 3+10 & 9+20 \end{pmatrix} = \begin{pmatrix} 9 & 19 \\ 13 & 29 \end{pmatrix}$$

$$B' = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix}$$

$$\times \quad A'B' = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1+9 & 4+15 \\ 2+12 & 8+20 \end{pmatrix} = \begin{pmatrix} 10 & 19 \\ 14 & 28 \end{pmatrix}$$

5. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad A' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = A$

Determinants

$$|A|_{2\times 2} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \underline{a_{11}a_{22}} - \underline{a_{12}a_{21}}$$

e.g. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 3\cdot 2 = \boxed{-2}$

1. Select one row or one column.

$$|A|_{3\times 3} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

a_{ij} i = row (0 th), j = 0 th column

$i+j$ = even number $\oplus$ $i+j$ = odd number $\ominus$

$$= \oplus \boxed{a_{11}} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \ominus a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \oplus \boxed{a_{13}} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$1+1=2$ $1+2 = 3$ odd $1+3=4$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = \begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{vmatrix}$$

e.g. $|A|_{3\times 3} = \begin{vmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{vmatrix} = +1 \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} - 0 \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix}$

ij $1+1$

$$= 1(1) - 0 + 2(-3-2)$$
$$= 1 - 10 = -9 \;//$$

$$= 2\begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 2(-3-2) + 1(1)$$
$$= -10 + 1 = -9.$$

$$= 2\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 2(-2) - 3(2) + 1(1)$$
$$= -4 - 6 + 1 = -9 \;//$$

$$\underset{4\times 4}{A} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 4 & 2 & 1 & 0 \end{vmatrix} = \ominus 2 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 4 & 2 & 1 \end{vmatrix}$$

a_{12} a_{14}

R-practice

a_{ij} $a_{\boxed{23}} \Rightarrow 2+3 = 5$ odd

$$= -2\left(1\cdot(-2) + 1\cdot(2-4)\right) + 1\left(1\cdot(1-6) + 1(2-4)\right)$$

$$=$$

nth order determinants

- Minor of the element a_{11} $= |M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$\underset{3\times 3}{|A|} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}|M_{11}| \underbrace{- a_{12}|M_{12}|}_{|C_{12}|} + a_{13}|M_{13}|$$

$$|M_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad |M_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$|A| = a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}|$

- Cofactor $|C_{ij}| \equiv (-1)^{i+j} |M_{ij}|$

if $i+j$ = even $\Rightarrow$ e.g. $(-1)^2 \Rightarrow (+)$

if $i+j$ = odd $\Rightarrow$ e.g. $(-1)^3 \Rightarrow (-)$

e.g. $|A| = \begin{vmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$

minor a_{11}

$|M_{11}| = \begin{vmatrix} 5 & 4 \\ 2 & 1 \end{vmatrix} = 5-8 = -3 \quad |C_{11}| = (-1)^{1+1}(-3) = -3$

$|M_{12}| = \begin{vmatrix} 6 & 4 \\ 3 & 1 \end{vmatrix} = 6-12 = -6 \quad |C_{12}| = (-1)^{1+2}(-6) = 6$

$|M_{13}| = \begin{vmatrix} 6 & 5 \\ 3 & 2 \end{vmatrix} = 12-15 = -3 \quad |C_{13}| = (-1)^{1+3}(-3) = -3$

$|A| = 9(-3) + 8(6) + 7(-3) =$

3rd order determinant

$|A| = a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| = \sum_{j=1}^{3} a_{ij} |C_{ij}| \leftarrow$ 1st row

4th order det.

$\underset{4\times 4}{|A|} = a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| + a_{14}|C_{14}|$

nth order det.

$\underset{n\times n}{|A|} = a_{11}|C_{11}| \cdots\cdots a_{1n}|C_{1n}| = \sum_{j=1}^{n} a_{ij}|C_{ij}|$

e.g.

$$\underset{4\times 4}{|A|} = \begin{vmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & \underset{a_{23}}{4} & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & 8 \end{vmatrix} = -4 \underset{3\times 3}{\begin{vmatrix} 1 & 2 & 9 \\ 1 & 6 & -1 \\ 0 & -5 & 8 \end{vmatrix}} = -4\left[1\underset{2\times 2}{(\underset{48}{6\cdot 8} - 5)} - 1\underset{2\times 2}{(\underset{16+45}{2\cdot 8 + 9\cdot 5})}\right]$$

$= -4(43-61)$

$= 72$ //