e.g. $Z = 4x^2 + 3xy + 6y^2$, $s.t.\ x + y = 56$ Check the 2nd order condition.

$\mathcal{L} = 4x^2 + 3xy + 6y^2 - \lambda(x + y - 56)$

$\mathcal{L}'_x = 8x + 3y - \lambda = 0$

$\mathcal{L}'_y = 3x + 12y - \lambda = 0$

$8x + 3y = 3x + 12y$

$5x = 9y$

$x = \frac{9}{5}y$

$\mathcal{L}'_\lambda = -x - y + 56 = 0$

$\frac{9}{5}y + y = 56$

$\frac{14}{5}y = 56$

$y^* = 56\left(\frac{5}{14}\right) = 20$

$x^* = 36$

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 12 \end{vmatrix} = -(12-3) + (3-8) = -9 - 5 < 0$$

minimization

e.g. $\min C(K, L) = 8K + 5L \quad s.t.\ 10K^{1/2}L^{1/3} = 200$

$\mathcal{L} = 8K + 5L - \lambda(10K^{1/2}L^{1/3} - 200)$

$L'_K = 8 - 5\lambda K^{-1/2}L^{1/3} = 0$

$L'_L = 5 - \frac{10}{3}\lambda K^{1/2}L^{-2/3} = 0$

$L'_\lambda = -10K^{1/2}L^{1/3} + 200 = 0$

$\frac{8}{5} = \frac{3K^{-1/2}L^{1/3}}{2K^{1/2}L^{-2/3}} \Rightarrow \frac{8}{5} = \frac{3L}{2K}$

$16K = 15L$

$\boxed{K = \frac{15}{16}L}$

$10\left(\frac{15}{16}L\right)^{1/2} L^{1/3} = 10\frac{\sqrt{15}}{4}L^{\frac{5}{6}} = 9.68L^{5/6} = 200$

$L^{5/6} = 20.66 \rightarrow L^* = (20.66)^{6/5} = 37.86$

$K^* = \frac{15}{16}(37.86) = 35.49$

$\lambda K^{-1/2}L^{1/3} = \frac{8}{5}$

g'_K

$\lambda = \frac{8}{5}K^{1/2}L^{-1/3} = \frac{8}{5}(35.49)^{1/2}(37.86)^{-1/3}$

$= 2.84$

$\lambda^* = 2.84$

$$|\bar{H}| = \begin{array}{c} \\ g'_K \\ g'_L \end{array} \begin{vmatrix} 0 & 5K^{-1/2}L^{1/3} & \frac{10}{3}K^{1/2}L^{-2/3} \\ 5K^{-1/2}L^{1/3} & \frac{5}{2}\lambda K^{-3/2}L^{1/3} & -\frac{5}{3}\lambda K^{-1/2}L^{-2/3} \\ \frac{10}{3}K^{1/2}L^{-2/3} & -\frac{5}{3}\lambda K^{-1/2}L^{-2/3} & \frac{20}{9}\lambda K^{1/2}L^{-5/3} \end{vmatrix}$$

L''

$= -5K^{-1/2}L^{1/3}\left[(5K^{-1/2}L^{1/3})\left(\frac{20}{9}\lambda K^{1/2}L^{-5/3}\right) - \left(-\frac{5}{3}\lambda K^{-1/2}L^{-2/3}\right)\left(\frac{10}{3}K^{1/2}L^{-2/3}\right)\right]$

$+ \frac{10}{3}K^{1/2}L^{-2/3}\left[5K^{-1/2}L^{1/3}\left(-\frac{5}{3}\lambda K^{-1/2}L^{-2/3}\right) - \left(\frac{5}{2}\lambda K^{-3/2}L^{1/3}\right)\left(\frac{10}{3}K^{1/2}L^{-2/3}\right)\right]$

$K, L > 0$

< 0

$\boxed{M=1}$

< 0

minimization

<More Variables>

$\max f(x_1, \dots x_n)\ s.t.\ g(x_1, \dots x_n) = c$

e.g. max $x^2y^3z \quad s.t. \quad x + y + z = 12$

$\mathcal{L} = x^2y^3z - \lambda(x+y+z-12)$

$\mathcal{L}_x' = 2xy^3z - \lambda = 0$ ①

$\mathcal{L}_y' = 3x^2y^2z - \lambda = 0$ ②

$\mathcal{L}_z' = x^2y^3 - \lambda = 0$ ③

$\mathcal{L}_\lambda' = -x-y-z+12 = 0$ ④

$\frac{①}{②}\quad \frac{2xy^3z}{3x^2y^2z} = \frac{\lambda}{\lambda}$

$\frac{2}{3}\frac{y}{x} = 1 \Rightarrow \boxed{y = \frac{3}{2}x}$

$\frac{②}{③}\quad \frac{3x^2y^2z}{x^2y^3} = 1 \Rightarrow \frac{3z}{y} = 1$

$z = \frac{1}{3}y = \frac{1}{3}\left(\frac{3}{2}x\right) = \boxed{\frac{1}{2}x}$

Strategy: express y and z as functions of x, so that by using g(x,y,z), x can be determined.

$x, y \leftarrow z$

$x\ z \leftarrow y$

$x + \left(\frac{3}{2}x\right) + \frac{1}{2}x = 12$

$\Rightarrow x^* = 4,\ y^* = 6,\ z^* = 2$

S.O.C. for n variables, 1 constraint case

Maximization	Minimization
$\lvert\bar{H}_2\rvert > 0$	$\lvert\bar{H}_2\rvert < 0$
$(-1)^n\lvert\bar{H}_n\rvert > 0 \quad \lvert\bar{H}_3\rvert < 0\ (n=3)$	Always $< 0 \quad \lvert\bar{H}_3\rvert < 0$, $\lvert\bar{H}_4\rvert < 0$

$$\lvert\bar{H}\rvert = \begin{vmatrix} 0 & g_x & g_y & g_z & \cdots \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} & \mathcal{L}_{xz} & \cdots \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} & \mathcal{L}_{yz} & \cdots \\ g_z & \mathcal{L}_{zx} & \mathcal{L}_{zy} & \mathcal{L}_{zz} & \cdots \end{vmatrix}$$

$(n+1)\times(n+1)$

e.g. For the previous question (max $x^2y^3z \quad s.t. \quad x + y + z = 12$), check S.O.C.

$$\lvert\bar{H}\rvert = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2y^3z & 6xy^2z & 2xy^3 \\ 1 & 6xy^2z & 6x^2yz & 3x^2y^2 \\ 1 & 2xy^3 & 3x^2y^2 & 0 \end{vmatrix}$$

$\rightarrow \lvert\bar{H}_2\rvert = -1(6x^2yz - 6xy^2z) + (6xy^2z - 2y^3z)$

$= -(6xyz)(x - y) + (2y^2z)(3x - y)$

$x^* = 4,\ y^* = 6,\ z^* = 2$

$= \underline{1440} > 0$

$\rightarrow \lvert\bar{H}_4\rvert = -1985984 < 0$

More Constraints

$$\text{Max } f(x_1, \dots, x_2), st. \begin{cases} g_1(x_1, \dots, x_n) = c_1 \\ g_2(x_1, \dots, x_n) = c_2 \\ g_3(x_1, \dots, x_n) = c_3 \\ \vdots \end{cases}$$

$$L = f(x_1, \dots x_2) - \lambda_1 [g_1(x_1 \cdots x_n) - c_1] - \lambda_2 [g_2(x_1 \cdots x_n) - c_2] - \lambda_3 [g_3(x_1 \cdots x_n) - c_3]$$

$$= f(x_1 \cdots x_n) - \sum_{j=1}^{m} \lambda_j (g_j(x \cdots x_n) - c_j)$$

n: # of variables, $i = 1 \cdots n$

m: # of constraints, $j = 1 \cdots m$

FOC $$\frac{\partial L}{\partial x_i} = \frac{\partial f(x_1 \cdots x_n)}{\partial x_i} - \sum \lambda_j \frac{\partial g_j(x_1 \cdots x_n)}{\partial x_i} = 0 \qquad i = 1 \cdots n$$

e.g. $x^2 + y^2 + z^2 \quad st. x + 2y + z = 30, \quad 2x - y - 3z = 10$. Find x*, y* and z*.

$$L = x^2 + y^2 + z^2 - \lambda_1(x + 2y + z - 30) - \lambda_2(2x - y - 3z - 10)$$

FOC

$L'_x = 2x - \lambda_1 - 2\lambda_2 = 0$ ①

$L'_y = 2y - 2\lambda_1 + \lambda_2 = 0$ ②

$L'_z = 2z - \lambda_1 + 3\lambda_2 = 0$ ③

$L'_{\lambda_1} = -x - 2y - z + 30 = 0$ ④

$L'_{\lambda_2} = -2x + y + 3z + 10 = 0$ ⑤

$\lambda_1 = 2x - 2\lambda_2$

$2y - 2(2x - 2\lambda_2) + \lambda_2 = 0$

$5\lambda_2 = 4x - 2y$

$\lambda_2 = \frac{4}{5}x - \frac{2}{5}y$ ⑥

$\lambda_1 = 2x - 2\left(\frac{4}{5}x - \frac{2}{5}y\right)$

$= \frac{2}{5}x + \frac{4}{5}y$ ⑦

$2z - \left(\frac{2}{5}x + \frac{4}{5}y\right) + 3\left(\frac{4}{5}x - \frac{2}{5}y\right) = 0$

$2z = -\frac{10}{5}x + \frac{10}{5}y$

$z = -x + y$ ⑧

$-x - 2y - (-x + y) + 30 = 0$

$3y = 30 \Rightarrow y^* = 10$ ⑨

⑧, ⑨ into ⑤

$-2x + 10 + 3(-x + 10) + 10 = 0$

$-5x + 50 = 0$

$x^* = 10$ ⑩

$z^* = -10 + 10 = 0$ ⑪

$$\begin{cases} x^* = 10 \\ y^* = 10 \\ z^* = 0 \end{cases}$$

<S.O.C for n variables, m constraints case>

1st constraint, 1st derivative w.r.t. 1st var.

$m \times n$

$m \times m$

$$|\bar{H}| = \left|\begin{array}{ccc|cccc} 0 & \cdots & 0 & g^1_1 & g^1_2 & \cdots & g^1_n \\ \vdots & \ddots & & g^2_1 & g^2_2 & \cdots & g^2_n \\ & & & \vdots & & & \\ 0 & & 0 & g^m_1 & g^m_2 & \cdots & g^m_n \\ \hline g^1_1 & g^2_1 & \cdots\ g^m_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g^1_2 & g^2_2 & \cdots\ g^m_2 & L_{21} & L_{22} & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ g^1_n & g^2_n & \cdots\ g^m_n & L_{n1} & & & L_{nn} \end{array}\right|$$

$(m+n) \times (m+n)$

$n \times m$

$n \times n$

m constraints

n variables

$|\bar{H}_2|$ =

$$|\bar{H}_2| = \left|\begin{array}{ccc|cc} & & & g^1_1 & g^1_2 \\ & 0 & & \vdots & \vdots \\ & & & g^m_1 & g^m_2 \\ \hline g^1_1 & \cdots & g^m_1 & L_{11} & L_{12} \\ g^1_2 & \cdots & g^m_2 & L_{21} & L_{22} \end{array}\right|$$

$|\bar{H}_3| =$

$|\bar{H}_1|$
$(-1)^1 < 0$
$m = 0$ $\rightarrow$

General Rule

Maximization	Minimization
$\lvert\bar{H}_{m+1}\rvert$ $(-1)^{m+1}$	$\lvert\bar{H}_{m+1}\rvert$ $(-1)^{m}$

e.g. $|\bar{H}_2| > 0$
$m=1$ $(-1)^2$

$m = 2$ $|\bar{H}_3| < 0$
$(-1)^3$

m = odd ≤ 0

m = even ≥ 0

e.g.For the previous question ($x^2+y^2+z^2$ $\ st.\ x+2y+z=30$, $\ 2x-y-3z=10$) , check S.O.C.

$n=3,\ m=2$

$$|\bar{H}| = \begin{vmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 2 & 0 \\ 1 & -3 & 0 & 0 & 2 \end{vmatrix}$$

Derivati of FOC

$m=2$

$|\bar{H}_{m+1}| = |\bar{H}_3|$

if max $(-1)^{m+1} < 0$

if min $(-1)^m > 0$

$= 150 > 0 \rightarrow f(x,y,z)$ is minimized at $(10,10,0)$

)-

<Envelope Theorem>

for constrainted optimization (n=2, m=1 case)

$$\max f(x_1,x_2)\ \ st.\ g(x_1,x_2)=c.$$

1000

$$\Rightarrow L = f(x_1,x_2) - \lambda(g(x_1,x_2) - c)$$

Question: When c goes up by 1 unit, how much the optimal value of f* changes?

Answer: Long way => 1. Derive value function, 2. Then find $\frac{df^*(c)}{dc}$

$P_1X_1 + P_2X_2 = M$

Short way (Envelope Thoerem) => 1. $\frac{dL}{dc}\Big|_{x=x^*(c)}$ (Take the derivative of L w.r.t. c directly, then evaluate at the optimal value of x)

$M = 1000 \rightarrow U^* = 100$

$M = 1001 \rightarrow \Delta U^*?$

$$\boxed{\frac{\partial f^*(c)}{\partial c} = \frac{\partial L(x,c)}{\partial c}\Big|_{x=x^*(c)} = \lambda^*(c)}$$

⇨ More generally, (n variables, m constraints)

$$\max f(x_1 \cdots x_n) \quad \text{s.t.} \begin{cases} g_1(x_1 \cdots x_n) = c_1 \\ \vdots \\ g_m(x_1, \cdots x_n) = c_m \end{cases}$$

$$\frac{\partial f^*(c)}{\partial c_j} = \frac{\partial L(x,c)}{\partial c_j}\Big|_{x=x^*(c)} = \lambda_j^*$$

$n=3, m=1$

e.g. $x+4y+3z, st. x^2+2y^2+\frac{1}{3}z^2=c$. Find x*, y*, z*. When c goes up by 1 unit, how much f* will change? (n=3, m=1 case)

$(x, y, z > 0)$

$$L = x+4y+3z - \lambda\left(x^2+2y^2+\frac{1}{3}z^2 - c\right)$$

$$\frac{\partial L}{\partial c}\Big|_{x=x^*(c)} = \lambda^* = \frac{1}{2x^*} = \frac{1}{2}\frac{6}{\sqrt{c}} = \frac{3}{\sqrt{c}}$$

$c \uparrow$ by 1 unit, $f^* \uparrow$ by $\frac{3}{\sqrt{c}}$

$$\begin{cases} L_x' = 1-2\lambda x = 0 & ① \rightarrow \lambda = \frac{1}{2x} \\ L_y' = 4-4\lambda y = 0 & ② \rightarrow \lambda = \frac{1}{y} \\ L_z' = 3-\frac{2}{3}\lambda z = 0 & ③ \rightarrow \lambda = \frac{9}{2z} \\ L_\lambda' = -x^2-2y^2-\frac{1}{3}z^2+c=0 & ④ \end{cases}$$

$y=2x$

$z=9x$

$$x^2+2(2x)^2+\frac{1}{3}(9x)^2=c$$

$$\left(1+8+\frac{81}{3}\right)x^2=c$$

$$x^2=\frac{c}{36}$$

$$\boxed{x^*=\frac{\sqrt{c}}{6}} \quad y^*=\frac{\sqrt{c}}{3}, \quad z^*=\frac{3}{2}\sqrt{c}$$

$$\rightarrow f^*(x^*(c), y^*(c), z^*(c)) = f^*(c) \Rightarrow \frac{\partial f^*}{\partial c}$$

e.g. $\max U(x_1, \ldots, x_n)$ $st. p_1 x_1 + \cdots + p_n x_n = m$.

$$\mathcal{L}(\underbrace{x_1, \cdots x_n}_{\text{variable}} ; \underbrace{p_1 \cdots p_n, m}_{\text{parameter}}) = U(x_1 \cdots x_n) - \lambda(p_1 x_1 + \cdots + p_n x_n - m)$$

From Envelope Theorem,

$$\frac{\partial U^*}{\partial m} = \frac{\partial \mathcal{L}(x_1^* \cdots x_n^* ; p_1, \cdots m)}{\partial m} = \lambda^*$$

As income (m) goes up by 1 unit, maximized utility will go up by λ^*

$$\frac{\partial U^*}{\partial p_1} = \frac{\partial \mathcal{L}(x_1^* \cdots x_n^* ; p_1, \cdots m)}{\partial p_1} = -\lambda^* x_1^* \quad \text{(Roy's Identity)}$$

As the price of good 1 (p1) goes up by 1 unit, the maximized utility will go down by $\lambda^* x_1^*$

<Optimization Problem with Inequality Constraint> $10X_1 + 20X_2 = 100$

$$\max f(x,y) \quad st.\, g(x,y) \leq c$$

$$L = f(x,y) - \lambda[g(x,y) - c]$$

Kuhn-Tucker Necessary Conditions

→ ① $L'_x = f'_x - \lambda g'_x = 0$

→ ② $L'_y = f'_y - \lambda g'_y = 0$

③ $\lambda \geq 0$, $g(x,y) \leq c$ & $\boxed{\lambda[g(x,y) - c] = 0}$

$\Rightarrow$ $\boxed{\lambda = 0}$, $g(x,y) < c$ OR $g(x,y) = c$, $\lambda > 0$

Case 1 $\lambda > 0$ *if* $g(x,y) = c$. (*Same as equality constraint*)

→ Case 2 (NEW!) $\boxed{\lambda = 0}$ *if* $g(x,y) < c$ (<= No all resources are used). λ : *inactive slack*

e.g. $f(x,y) = x^2 + y^2 + y + 1$, $st.\, x^2 + y^2 \leq 1$

$$L = x^2 + y^2 + y + 1 - \lambda(x^2 + y^2 - 1)$$

→ ① $L'_x = 2x - 2\lambda x = 0$

→ ② $L'_y = 2y + 1 - 2\lambda y = 0$

→ ③ $\lambda \geq 0$, $x^2 + y^2 \leq 1$, $\boxed{\lambda(x^2 + y^2 - 1) = 0}$

→ $\boxed{\text{Case 1}}$ $\lambda > 0$, $x^2 + y^2 = 1$ ④

→ $\boxed{\text{Case 2}}$ $\lambda = 0$, $x^2 + y^2 < 1$ ⑤

From ① $2x(1 - \lambda) = 0$ → $x = 0$ (case A); $\lambda = 1$ (case B)

If case B, $\lambda = 1$ → ② $2y + 1 = 2y$ ← contradiction → $\lambda \neq 1$

→ case A + ④ $x^2 + y^2 = 1$ ($x = 0$) → $y^2 = 1$ ⇒ $y = \pm 1$ → $(0, 1)$, $(0, -1)$

$f(0,1) = 1 + 1 + 1 = \boxed{3}$ MAX

$f(0,-1) = 1 - 1 + 1 = 1$

From ⑤ $y^2 < 1$ → $\underbrace{-1 < y < 1}$ then $\lambda = 0$ → ② $2y + 1 = 0$, $y = -\frac{1}{2}$

$f(0, -\frac{1}{2}) = (-\frac{1}{2})^2 + (-\frac{1}{2}) + 1 = -\frac{5}{4}$ min