ECO106/138 Week 11 Lecture Note Template

Differentiating Systems of Equations

How can we find the partial derivative of the implicit functions?

e.g. $3x_1 + x_2^2 - y_1 - 3y_2^3 = 0$

$x_1^3 - 2x_2 + 2y_1^3 - y_2 = 0$

$n = 4$
$m = 2$
$n - m = 2 > 0$

Find DF, $\frac{dy_1}{dx_1}, \frac{dy_2}{dx_1}$ → express $y_1 = f(x_1 \dots)$

→ $3dx_1 + 2x_2 dx_2 - dy_1 - 9y_2^2 dy_2 = 0$

→ $3x_1^2 dx_1 - 2dx_2 + 6y_1^2 dy_1 - dy_2 = 0$

Solve for dy_1, dy_2 in terms of dx_1 & dx_2

$dy_1 = 3dx_1 + 2x_2 dx_2 - 9y_2^2 \overbrace{(3x_1^2 dx_1 - 2dx_2 + 6y_1^2 dy_1)}^{dy_2}$

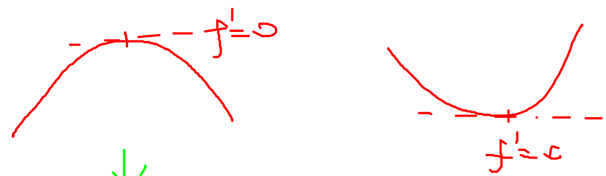
$(1 + 54y_1^2 y_2^2) dy_1 = 3dx_1 + 2x_2 dx_2 - 27x_1^2 y_2^2 dx_1 + 18y_2^2 dx_2$

$= (3 - 27x_1^2 y_2^2) dx_1 + (2x_2 + 18y_2^2) dx_2$

$$\frac{dy_1}{dx_1} = \frac{3 - 27x_1^2 y_2^2}{1 + 54y_1^2 y_2^2}$$

Try & find out

$$\frac{dy_2}{dx_1} = \frac{3x_1^2 + 18y_1^2}{1 + 54y_1^2 y_2^2}$$



Multivariable Optimization

Conditions for local extrema

Conditions	Maximum	Minimum
First-order necessary condition	$f_1'(x,y) = f_2'(x,y) = 0$	$f_1'(x,y) = f_2'(x,y) = 0$
Second-order sufficient condition	$f_{11}''(x,y), f_{22}''(x,y) < 0$	$f_{11}''(x,y), f_{22}''(x,y) > 0$
	$D = f_{11}''(x,y)f_{22}''(x,y) - ((f_{12}''(x,y))^2 \geq 0$ (determinant of Hessian Matrix)	

$f'(\ ) = 0 \rightarrow (a,b)$

case1 if $D>0$, $f_{11}'', f_{22}'' < 0$
(a,b) is a local maximum

$$H = \begin{bmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{bmatrix} \Rightarrow |H| = f_{11}''f_{22}'' - f_{12}''f_{21}'' = f_{11}''f_{22}'' - (f_{12}'')^2$$

case2 if $D>0$, $f_{11}'', f_{22}'' > 0$
(a,b) is a local min.

case3 if $D<0$, (a,b) is *a saddle point*

case4 if $D=0$, *second derivative test is inconclusive.* → can be min/max/saddle point.

e.g. $f(x,y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$. Find an extreme point and confirm if it's max/min.

FOC SOC

e.g. x, y inputs
f output.

F.O.C. $\rightarrow f_x' = -4x - 2y + 36 = 0$
$\rightarrow f_y' = -2x - 4y + 42 = 0$

$-4x - 2y + 36 = 0$
$*(-2)$ $4x + 8y - 84 = 0$
$6y - 48 = 0$
$y = 8$
$4x = -16 + 36 = 20$
$x = 5$

$f(5,8) = -2(5^2) - 2(5)(8) - 2(8^2) + 36(5) + 42(8) - 158$
$= 100$

SOC. $f_{xx}'' = -4 < 0$, $f_{yy}'' = -4 < 0$, $f_{xy}'' = -2$

$D = (-4)(-4) - (-2)^2 = 16 - 4 = 12 > 0$

→ at $(5,8)$ f is maximized
maximized value is 100.

TR − TC

$K, L \geq 0$

e.g. $\pi(K,L) = 12K^{1/2}L^{1/4} - 1.2K - 0.6L$ Find the extreme point and confirm if it's max/min.

$\underbrace{K^{1/2}L^{1/4}}_{PQ}$ $\underbrace{1.2K - 0.6L}_{Cost}$

FOC $\rightarrow \pi_K' = 6K^{-1/2}L^{1/4} - 1.2 = 0$ ①
$\rightarrow \pi_L' = 3K^{1/2}L^{-3/4} - 0.6 = 0$ ②

$\Rightarrow \dfrac{6K^{-1/2}L^{1/4}}{3K^{1/2}L^{-3/4}} = \dfrac{1.2}{0.6} \Rightarrow 2\dfrac{L}{K} = 2 \Rightarrow L = K$

$K = L \Rightarrow 6K^{-1/2}K^{1/4} = 1.2$
$K^{-1/4} = 0.2$
$K^* = (0.2)^{-4} = 625 = L^*$

$\pi(625, 625) = 12(625)^{1/2}(625)^{1/4} = 375$ $\boxed{\text{maximized}}$?

SOC $\pi_{KK}'' = -3K^{-3/2}L^{1/4} < 0$
$\pi_{LL}'' = -\frac{9}{4}K^{1/2}L^{-7/4} < 0$

π_{KL}''

$D = \left(-3K^{-3/2}L^{1/4}\right)\left(-\frac{9}{4}K^{1/2}L^{-7/4}\right) - \left(\frac{3}{2}K^{-1/2}L^{-3/4}\right)^2$
$= \frac{27}{4}K^{-1}L^{-3/2} - \frac{9}{4}K^{-1}L^{-3/2} > 0$
$= \frac{18}{4}K^{-1}L^{-3/2} > 0$ ✓

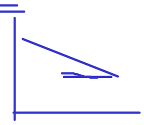
π is maximized at $(625, 625)$
∴ max π = 375 //

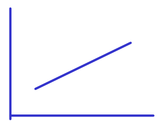
Linear Models with Quadratic Objectives (An Economic Example)

Domestic EU

Suppose a firm is producing good Q and selling in market 1 and 2. Given the following demand and cost functions, find the profit maximizing level of Q1 and Q2 and the maximized profit.

Demand Functions





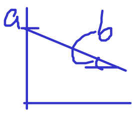
$$\begin{cases} p_1 = a_1 - b_1 Q_1 \\ p_2 = a_2 - b_2 Q_2 \end{cases}$$

$b_1 > 0$, $b_2 > 0$

Q_1, Q_2

P_1, P_2

coefficients: a_1, a_2, b_1, b_2, α



Cost Function

$$C(Q) = \alpha(Q_1 + Q_2)$$

Objective fn

$P \cdot Q$

TR − TC

$TR_1 + TR_2$

$$\rightarrow \pi(Q_1, Q_2) = (a_1 - b_1 Q_1) Q_1 + (a_2 - b_2 Q_2) Q_2 - \alpha(Q_1 + Q_2)$$

P_1 $\quad$ P_2

FOC. $\pi'_{Q_1} = a_1 - 2b_1 Q_1 - \alpha = 0 \Rightarrow Q_1^* = \dfrac{a_1 - \alpha}{2b_1}$

$\pi'_{Q_2} = a_2 - 2b_2 Q_2 - \alpha = 0 \Rightarrow Q_2^* = \dfrac{a_2 - \alpha}{2b_2}$

$$\Rightarrow \pi^*\left(\frac{a_1 - \alpha}{2b_1}, \frac{a_2 - \alpha}{2b_2}\right) = \frac{(a_1 - \alpha)^2}{4b_1} + \frac{(a_2 - \alpha)^2}{4b_2}$$

confirm D

SOC $\pi''_{11} = -2b_1 < 0$ since $b_1 > 0$, $b_2 > 0$ slope of demand fn.

$\pi''_{22} = -2b_2 < 0$

$$D = \pi''_{11}\pi''_{22} - (\pi''_{12})^2 = (-2b_1)(-2b_2) - (0) = 4b_1b_2 > 0 \quad \checkmark$$

$$\boxed{\pi''_{12} = \pi''_{21}}$$

e.g. Solve the previous question with specific parameters:

Demand Functions

$$\overset{a_1}{} \quad \overset{b_1}{}$$

$$p_1 = 100 - Q_1$$

$$p_2 = 80 - Q_2$$

$$\underset{a_2}{} \quad \underset{b_2}{}$$

Cost Function

$$C(Q) = \underset{\alpha}{6}(Q_1 + Q_2)$$

① Use the result of previous question

② Solve the problem from the beginning

$$\Pi = (100 - Q_1)Q_1 + (80 - Q_2)Q_2 - 6(Q_1 + Q_2)$$

$$Q_1^* = \frac{a_1 - \alpha}{2b_1} = \frac{100-6}{2(1)} = 47$$

$$Q_2^* = \frac{a_2 - \alpha}{2b_2} = \frac{80-6}{2} = 37$$

$$\Pi^* = \frac{(100-6)^2}{4(1)} + \frac{(80-6)^2}{4(1)} = \underline{\underline{3578}}$$

maximized Π

e.g. If it's illegal to price discriminate, how much profit will be lost?

no regulation

$$\Pi^* = 3578$$

$$\Pi^*_{w/} = \boxed{}$$

Demand Functions

$$p = 100 - Q_1$$

$$p = 80 - Q_2$$

→ Find p → P.Q

$$\rightarrow Q_1 = 100 - p, \quad Q_2 = 80 - p \rightarrow Q = Q_1 + Q_2 = 180 - 2p$$

$$2p = 180 - Q$$

$$\boxed{p = 90 - \tfrac{1}{2}Q}$$

Cost Function

$$C(Q) = 6(Q_1 + Q_2)$$

$$\Pi = \left(90 - \tfrac{1}{2}Q\right)Q - 6Q$$

$$\frac{\partial \Pi}{\partial Q} = 90 - Q - 6 = 0 \Rightarrow Q^* = 84$$

$$\Pi = \left(90 - \tfrac{1}{2}(84)\right)(84) - 6(84) = \underline{\underline{3528}}$$

$$\frac{\partial^2 \Pi}{\partial Q^2} = -1 < 0$$

$$\text{loss} = 3578 - 3528 = \underline{\underline{50}}$$

Three or more variables

3 variables

$$z = f(x_1, x_2, x_3)$$

$$FOC: f_1' = f_2' = f_3' = 0$$

$$SOC: |H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = f_{11}\begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix} - f_{12}\begin{vmatrix} f_{21} & f_{23} \\ f_{31} & f_{33} \end{vmatrix} + f_{13}\begin{vmatrix} f_{21} & f_{22} \\ f_{31} & f_{32} \end{vmatrix}$$

$|H_1| = |f_{11}|$

$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11}f_{22} - f_{12}f_{21}$

$|H_3| = |H|$

For Maximum	For Minimum
$\lvert H_1\rvert < 0$ ✓ → $\lvert H_2\rvert > 0$ ✓ $\lvert H_3\rvert < 0$ ✓ $(-1)^n$ ← the order of determinant.	$\lvert H_1\rvert > 0$ → $\lvert H_2\rvert > 0$ $\lvert H_3\rvert > 0$ } all positive

e.g. $f(x,y,z) = 2x - x^2 + 10y - y^2 + 3 - z^2$. Find the extreme value and classify the point as max/min.

FOC → $f_x' = 2 - 2x = 0$

$f_y' = 10 - 2y = 0$

→ $f_z' = -2z = 0$

→ $\begin{cases} x = 1 \\ y = 5 \\ z = 0 \end{cases}$

$(1, 5, 0)$

S.O.C

$H =$ $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ f_{xx}'' f_{yy}'' f_{zz}''

Hessian.

$-2\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}$

$f(1,5,0) = 29$

$|H_1| = -2 < 0$.

$|H_2| = 4 > 0$.

$|H_3| = -2(4) = -8 < 0$.

→ MAX

N variables case $z = f(x_1, x_2 \cdots x_n)$

FOC $f_1' = f_2' = \cdots = f_n' = 0$

$$H_{n\times n} = \begin{bmatrix} f_{11}'' & f_{12}'' & \cdots & f_{1n}'' \\ \vdots & & \ddots & \\ f_{n1}'' & & \cdots & f_{nn}'' \end{bmatrix}$$

For Maximum	For Minimum
$\lvert H_1\rvert < 0$ $\lvert H_2\rvert > 0$ $\quad (-1)^n \lvert H_n\rvert > 0$ $\lvert H_3\rvert < 0$ $\vdots$ n : odd $\quad$ n=odd $\quad$ n = even $\lvert H_n\rvert < 0$ $\quad > 0$	$\lvert H_1\rvert > 0$ $\vdots$ all positive. $\lvert H_n\rvert > 0$

Comparative Statics and Envelope Theorem.

Analyze what's happen to the optimal solution when other parameters change.

start with an objective funcion
f(x(r),r) : x=variable, r=parameter/coefficient/constant

max f(x(r),r)), with respect to (wrt) x (by keeping r constant).

FOC f'x(x(r),r)=0 => we can find x which maximize f(x(r),r) (objective function).*

Value function = objective function evaluated at the optimal
= f(x*(r),r) = f*(r).*

If we are interested in "what's happen if r changes at the optimal?" "how much maximized value of f will change if r changes by one unit?"

$$\frac{\partial f^*(x^*(r),r)}{\partial r} = \underbrace{f_{x^*}'(x^*(r),r)}_{=0}\frac{\partial x^*}{\partial r} + f_r'(x^*(r),r)$$

F.O.C

$$= f_r^*(x(r),r) = \frac{\partial f^*}{\partial r}$$

Ex.

e.g. $R(x) = rx, c(x) = x^2, r > 0$. Illustrate the envelope theorem.

obj. $\boxed{\Pi = rx - x^2}$

① Find the profit MAX. level of x
then derive Value fn.

③ If Δr by 1 unit, what's happen to Π^*?

② Envelope

$\frac{\partial \Pi}{\partial x} = r - 2x = 0$

$x^* = \frac{r}{2}$

$\Pi^* = r(x^*) - (x^*)^2 = r\left(\frac{r}{2}\right) - \left(\frac{r}{2}\right)^2 = \frac{r^2}{2} - \frac{r^2}{4}$

$= \frac{r^2}{4}$ value fn

non-envelope Theory way

① $\Rightarrow \frac{\partial \Pi^*}{\partial r} = \frac{2r}{4} = \frac{r}{2}$ //

$\Rightarrow \left.\frac{\partial f}{\partial r}\right|_{x=x^*}$

Take derivative of objective fn wrt r, evaluate at x^*

W/ En.

③ $\left.\frac{\partial \Pi}{\partial r}\right|_{x^*} = x^* = \frac{r}{2}$

Long way: solve the optimization, find x, find value function, then take the derivative of value function wrt r.*

Short way: with Envelope theorem take the derivative of f (objective function) wrt r. evaluate at x = x.*

e.g. $\pi = pf(x) - wx. \, Illustarte\ the\ envelope\ theorem.$ (How x*(w,p) will change according to p?)

e.g. $\pi = p\left(K^{2/3} + L^{1/2} + T^{1/3}\right) - rK - wL - qT$. Find K*, L* and T*, then find $\frac{d\pi^*}{dr}, \frac{d\pi^*}{dw}, \frac{d\pi^*}{dq}$.

Envelope Thm

$$\left.\frac{\partial \pi}{\partial r}\right|_{K=K^*} = -K^* = -\left(\frac{2}{3}\frac{p}{r}\right)^3$$

$$\pi'_K = \frac{2}{3} p K^{-1/3} - r = 0$$

$$K^* = \left(\frac{3}{2}\frac{r}{p}\right)^{-3} = \left(\frac{2}{3}\frac{p}{r}\right)^3$$

$$\pi'_L = \frac{1}{2} p L^{-1/2} - w = 0$$

$$\left.\frac{\partial \pi}{\partial w}\right|_{L=L^*} = -L^* = -\left(\frac{p}{2w}\right)^2$$

$$L^* = \left(2\frac{w}{p}\right)^{-2} = \left(\frac{p}{2w}\right)^2$$

$$\left.\frac{\partial \pi}{\partial q}\right|_{T=T^*} = -T^* = -\left(\frac{3q}{p}\right)^{-3/2}$$

$$\pi'_T = \frac{1}{3} p T^{-2/3} - q = 0$$

$$T^* = \left(\frac{3q}{p}\right)^{-3/2}$$

//