ECO106/138 Week 10 Lecture Note Template

<Euler's Theorem>

$f(tx_1, tx_2) = t^k f(x_1, x_2)$

$f(x_1 \cdots x_n)$ HD k

The differentiable function f of n variables is homogeneous of degree k if and only if

$$\boxed{\sum_{i=1}^{n} x_i f_i'(x_1 \cdots x_n) = k f(x_1, \cdots x_n)}$$

$x_1 f_1'(x_1 \cdots x_n) + x_2 f_2'(x_1 \cdots x_n) + \cdots x_n f_n'(x_1 \cdots x_n)$

Derivation:

$f(tx_1, tx_2) = t^k f(x_1, x_2)$

$\frac{d}{dt} \rightarrow x_1 f_1'(tx_1, tx_2) + x_2 f_2'(tx_1, tx_2) = k t^{k-1} f(x_1, x_2)$

$F = tx_1$, $\frac{dF}{dt} = x_1$ $\quad$ $F = tx_2$, $\frac{dF}{dt} = x_2$

set $t = 1$

$$\boxed{x_1 f_1'(x_1, x_2) + x_2 f_2'(x_1, x_2) = k f(x_1, x_2)}$$

LHS $\quad$ RHS

e.g. $Y = AK^{\alpha}L^{1-\alpha}$, check Euler's Theorem.

HD 1 $\quad$ We expect RHS $= 1\,Y = Y$

LHS $= AK^{\alpha}L^{1-\alpha}$?

$$LHS = K\frac{dY}{dK} + L\frac{dY}{dL} = K(\alpha AK^{\alpha-1}L^{1-\alpha}) + L((1-\alpha)AK^{\alpha}L^{-\alpha})$$

$$= \alpha AK^{\alpha}L^{1-\alpha} + (1-\alpha)AK^{\alpha}L^{1-\alpha}$$

$$= AK^{\alpha}L^{1-\alpha} = 1 \cdot Y = Y.$$

e.g. See if the following function is a homogenous function. If so, check Euler's Theorem.

$$f(x, y) = \frac{xy}{(x^2 + y^2)}$$

HD 0

$$f(tx, ty) = \frac{(tx)(ty)}{(tx)^2 + (ty)^2} = \frac{t^2(xy)}{t^2(x^2+y^2)} = t^0 \frac{xy}{x^2+y^2}$$

We expect RHS $= 0$?

$$x\frac{df}{dx} + y\frac{df}{dy} = x\left(\frac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2}\right) + y\left(\frac{x(x^2+y^2) - xy(2y)}{(x^2+y^2)^2}\right)$$

$$= \frac{\cancel{x^3y} + \cancel{xy^3} - \cancel{2x^3y} + \cancel{x^3y} + \cancel{xy^3} - \cancel{2xy^3}}{(x^2+y^2)^2}$$

$= 0$

4 properties of HD k functions

1. $f(x,y) is\ HDk \Leftrightarrow xf'_x(x,y) + yf'_y(x,y) = kf(x,y)$ <= *Euler's Theorem*

2. $f'_x(x,y)$ *and* $f'_y(x,y)$ *are* $HD(k-1)$

$\begin{cases} f(x,y) & HD\ k \\ f'(x,y) & HD(k-1) \end{cases}$

3. $f(x,y) = x^k f\left(1, \frac{y}{x}\right) = y^k f\left(\frac{x}{y}, 1\right)$

4. $x^2 f''_{xx}(x,y) + 2xy f''_{xy}(x,y) + y^2 f''_{yy}(x,y) = k(k-1)f(x,y)$

Derivation for 3.

$f(tx,ty) = t^k f(x,y) \leftarrow HD\ k$

Set $t = \frac{1}{x}$ $\Rightarrow f\left(1, \frac{y}{x}\right) = \left(\frac{1}{x}\right)^k f(x,y)$

$\Rightarrow x^k f\left(1, \frac{y}{x}\right) = f(x,y)$ *

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Set $t = \frac{1}{y}$ $\Rightarrow f\left(\frac{x}{y}, 1\right) = \left(\frac{1}{y}\right)^k f(x,y) \Rightarrow y^k f\left(\frac{x}{y}, 1\right) = f(x,y)$

Derivation for 4.

$\begin{cases} f(x,y) & \text{is } HD\ k \\ \Rightarrow^* f'_x(x,y) & HD(k-1) \\ f'_y(x,y) & HD(k-1) \end{cases}$

By Euler's Theorem,

$x f'_x(x,y) + y f'_y(x,y) = k f(x,y)$ ①

$x f''_{xx}(x,y) + y f''_{xy}(x,y) = (k-1) f'_x(x,y)$ ②

$x f''_{yx}(x,y) + y f''_{yy}(x,y) = (k-1) f'_y(x,y)$ ③

② * x

③ * y

$x^2 f''_{xx}(x,y) + xy f''_{xy}(x,y) = x(k-1) f'_x(x,y)$ ②'

$xy f''_{yx}(x,y) + y^2 f''_{yy}(x,y) = y(k-1) f'_y(x,y)$ ③'

②'+③'

$x^2 f''_{xx}(x,y) + 2xy f'_{xy}(x,y) + y^2 f''_{yy}(x,y)$

$= (k-1)\left[\underbrace{x f'_x(x,y) + y f'_y(x,y)}_{\shortparallel}\right]$

$= (k-1)\, k\, f(x,y)$

e.g. Given $f(x,y) = 3x^2y - y^3$, confirm 4 properties.

HD k ? $\boxed{HD3}$ $k=3$

4 properties of HD k functions

1. $f(x,y) is\ HDk \Leftrightarrow x f_x'(x,y) + y f_y'(x,y) = k f(x,y) \quad <= Euler's\ Theorem$

2. $f_x'(x,y)\ and\ f_y'(x,y)\ are\ HD(k-1)$

3. $f(x,y) \overset{①}{=} x^k f\left(1, \frac{y}{x}\right) \overset{②}{=} y^k f\left(\frac{x}{y}, 1\right)$

4. $\underbrace{x^2 f_{xx}''(x,y) + 2xy f_{xy}''(x,y) + y^2 f_{yy}''(x,y)}_{LHS} = \underbrace{k(k-1)f(x,y)}_{RHS}$ $6f(x,y)$

$\boxed{f(x,y) = 3x^2y - y^3}$

3. $x^3 f(1, y/x) = x^3\left(3\frac{y}{x} - \left(\frac{y}{x}\right)^3\right) = 3x^2y - y^3 = f(x,y)$

$y^3 f\left(\frac{x}{y}, 1\right) = y^3\left(3\left(\frac{x}{y}\right)^2 \cdot 1 - 1\right) = 3x^2y - y^3 = f(x,y)$

4. $LHS \overset{?}{=} 6(3x^2y - y^3) = \boxed{18x^2y - 6y^3}$ ← expectation

RHS

$f_x' = 6xy \qquad f_y' = 3x^2 - 3y^2$

$f_{xx}'' = 6y \qquad f_{yy}'' = -6y \qquad f_{xy}'' = 6x$

$x^2(6y) + 2xy(6x) + y^2(-6y) = 6x^2y + 12x^2y - 6y^3$

$= \boxed{18x^2y - 6y^3}$ LHS

✓

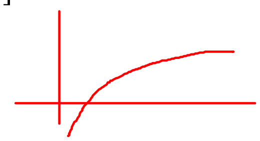
Homothetic Functions

A homothetic function is a monotonic transformation of a homogeneous function.

(monotonic transformation = a transformation by a strictly increasing function.)

$$H = h\big(Q(a,b)\big) \qquad [h'(Q) \neq 0]$$

Where Q(a,b) is HD k and h is strictly increasing function.



H = H(a,b) is not in general homogeneous.

e.g. $For\ Y = F(K,L) = \underbrace{K^{\alpha}L^{\beta}}_{HD\ \alpha+\beta}$, (a) see if it is a homogeneous function, (b) see if lnY is a homogeneous function.

$\ln(K^{\alpha}L^{\beta})$ = homothetic fn.

F: HD $\alpha+\beta$

$\ln F(K,L) = \ln(K^{\alpha}L^{\beta}) = \alpha \ln K + \beta \ln L$

HD ?

$\ln F(tK, tL) = \alpha \ln(tK) + \beta \ln(tL)$

$= \alpha \ln K + \beta \ln L + (\alpha+\beta)\ln t \neq t^{k} * F(K,L)$

$\alpha\beta > 0$ → Not Homogeneous function

e.g. $H = Q^2$, $Q = \underbrace{Ax^{\alpha}y^{\beta}}_{HD\alpha+\beta}$, see if H is a homothetic function. See if H is a homogeneous function.

Homothetic func

HD ?

$H = (Ax^{\alpha}y^{\beta})^2 = A^2x^{2\alpha}y^{2\beta}$

$H(tx, ty) = A^2(tx)^{2\alpha}(ty)^{2\beta} = t^{2\alpha+2\beta} * (A^2x^{2\alpha}y^{2\beta}) = (Ax^{\alpha}y^{\beta})^2$

HD $(2\alpha+2\beta)$ HD

HD $\alpha+\beta$

e.g. $H = e^{Q}$, $Q = Ax^{\alpha}y^{\beta}$, see if H is a homothetic function, see if H is a homogeneous function.

$H = e^{Ax^{\alpha}y^{\beta}}$

$H(tx, ty) = e^{A(tx)^{\alpha}(ty)^{\beta}} = e^{t^{\alpha+\beta}(Ax^{\alpha}y^{\beta})} \neq t^{k} H$

Not Homogeneous fn.

Linear Approximation

The linear approximation to f(x,y) about (x0,y0) is

$$f(x,y) \approx f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$

$x_0 = 2$

$y_0 = 3$

e.g. For $f(x,y) = xy^3 - 2x^3$, find f(2.01, 2.98) by using linear approximation about (2,3).

$f'_x = y^3 - 6x^2 \qquad f'_x(2,3) = 27 - 24 = 3$

$f'_y = 3xy^2 \qquad f'_y(2,3) = 3 \cdot 2 \cdot 9 = 54$

$f(2.01, 2.98) \approx f(2,3) + 3(2.01-2) + 54(2.98-3)$

$f(2,3) = 2(3^3) - 2(2)^3 = 38$

$= 38 + 3(0.01) + 54(-0.02) = 36.95$

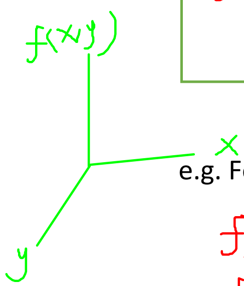
L.A

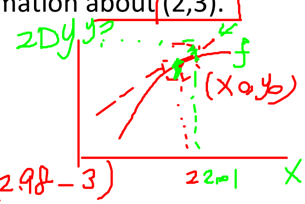
Approximated value

Actual value

$\Leftrightarrow f(2.01, 2.98) = (2.01)(2.98)^3 - 2(2.01)^3$

$= 36.9506\ldots$





Differentials

$z = f(x,y) \Rightarrow dz = f'_x(x,y)dx + f'_y(x,y)dx$ $\boxed{A}$

Rules $f(x,y), g(x,y)$

1. $d(af+bg) = adf + bdg$
2. $d(fg) = gdf + fdg$
3. $d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$, $g \neq 0$

e.g. Given $z = xy^2 + x^3$, find dz by using $\boxed{A}$ and 1.

$\boxed{A}$ $dz = (y^2 + 3x^2)dx + (2xy)dy$

① $dz = d(xy^2 + x^3) = (y^2 dx + 2xy\,dy) + 3x^2 dx$

$= (y^2 + 3x^2)dx + 2xy\,dy$

$f(x,y) = xy^2$

$g(x,y) = x^3$

e.g. Given $z = xe^{y^2}$, find dz.

$dz = e^{y^2}dx + 2yxe^{y^2}dy$

e.g. Given $z = \ln(x^2 - y^2)$, find dz.

$dz = \frac{2x}{x^2-y^2}dx - \frac{2y}{x^2-y^2}dy$

$= \frac{2x\,dx - 2y\,dy}{x^2-y^2}$

Systems of Equations

$$m \left\{ \begin{array}{l} f_1(\overbrace{x_1, x_2, \ldots, x_n}^{n}) = 0 \\ f_2(x_1, x_2, \ldots, x_n) = 0 \\ \ldots \\ f_m(x_1, x_2, \ldots, x_n) = 0 \end{array} \right.$$

var, var

$y = a + bX$

n: # of variables (a.k.a. total available degrees of freedom) $\Leftrightarrow$ constant term, coefficient

m: # of independent equations (constraints)

n-m: degrees of freedom <= # of variables which can be freely chosen.

- If n>m, n-m degrees of freedom
- If n<m, no solution to the system (inconsistent) ←
- If n=m, usually consistent, may not have unique solution.

e.g. (Try to) solve the following system of equations.

$$2x_1 + x_2 = -1 \qquad (1)$$

$$-3x_1 + x_2 = -2 \qquad (2)$$

$$-x_1 + x_2 = 1 \qquad (3)$$

$n = 2$

→ $m = 3$

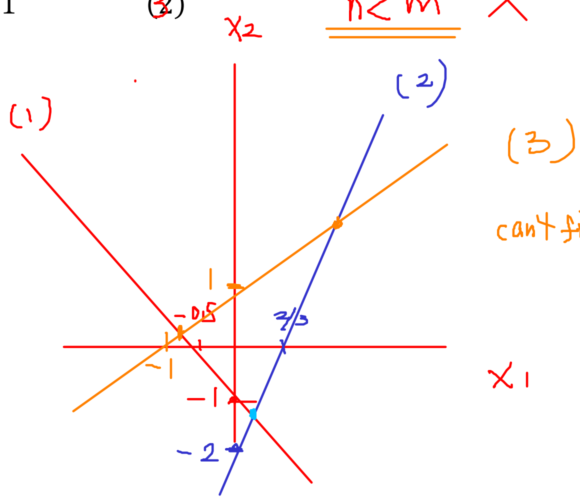
$n < m$ ✗

$n > m$

(1) $X_2 = -2X_1 - 1$

(2) $X_2 = 3X_1 - 2$

(3) $X_2 = X_1 + 1$



can't find a point satisfy all the eq.

e.g. Decide if the following system of equations is solvable.

$f(y+z+w)=x^3$ 1

$x^2+y^2+z^2=w^2$ 2

$g(x,y)-z^3=w^3$ 3

$n=4$ (x,y,z,w)

$m=3$

$4>$

$n>m$ ✓

$n-m=df=1$

once 1 variable is given, other three can be determined within the system.

e.g. Decide if the following system of equations is solvable.

$Y=C+I+G$

$C=f(Y-T)$

$I=h(r)$

$r=m(M)$

$n=7$ (Y,C,I,G,r,M,T)

$m=4$

$df=7-4=3$

usually G, T, M are exogenous

$\rightarrow$ $\boxed{Y, C, I, r}$ endogenous

Differentiating Systems of Equations

How can we find the partial derivative of the implicit functions?

HW

e.g. $3x_1+x_2^2-y_1-3y_2^3=0$

$x_1^3-2x_2+2y_1^3-y_2=0$

Find DF, $\frac{dy_1}{dx_1}, \frac{dy_2}{dx_1}$